

# Cosmological Constant, Classical "Vacuum" and Special Relativity

Yves Pierseaux (ypiersea@ulb.ac.be)

May 25, 2009

## Abstract

**VERSION 1 (results)** We show that Cosmological Constant  $\Lambda$  is not optional in GR (*general* relativity) because it is required by SR (*special* relativity). This completely unexpected result is obtained by introducing a minimal acceleration  $\alpha = \sqrt{\Lambda}$  (Milgrom) in Einstein boost with Lorentz Transformation (LT). We prove that hyperbolic rotation (LT) is an hyperbolic *motion* with a centrifugal acceleration. In Cosmological SR with  $\Lambda$  (CSR or CR), the universe is not only in expansion (with the law of Hubble) but even in accelerated expansion (cosmological parameter  $q = -1$ ). In CR the constant  $\Lambda$  is naturally associated to the bending of light (Ishak and Rindler). Given that the structure of space-time in Einstein's GR is determined by the *presence* of matter and  $\Lambda$  is associated to the *absence* of matter, we associate  $\Lambda$  not to "quantum" vacuum but to classical "vacuum" of Minkowski's space-time. Finally we show that 1917 Einstein's Cosmological Constant  $\Lambda$  corresponds to 1906 Poincaré's *non-electromagnetic* negative pressure and we deduce density of Poincaré's relativistic fluid.

## VERSION 2 (methodology)

In §1 we switch the scale radius factor  $a(t)$  of Friedman-Lemaître in the second member of Lorentz invariant  $x^2 - t^2 = a^2 = H^{-2} = \alpha_M^{-2} = \Lambda^{-1}$  ( $c = 1$ ,  $H$  is Hubble constant and  $\alpha_M$  is Milgrom's acceleration).

In §2 we sum up Einstein-Born-Rindler (EBR)'s model of acceleration in SR. EBR's rigid motion is induced from non-relativistic *rectilinear* uniformly accelerated motion.

In §3 we propose another relativistic theory of acceleration induced from non-relativistic uniformly *circular* accelerated motion given that Euclidean rotation (centripetal acceleration) must be replaced by Hyperbolic Rotation *motion* (*centrifugal* acceleration). This new theory of acceleration in Lorentz boost involves not a rigid motion but an elastic expanding motion. We introduce a *new symmetry* in Minkowski's basic diagram of (scale) hyperbolas by adopting a new definition of invariant proper distance in such a way that the LTed proper distance be a dilated distance (symmetrically with invariant proper time). By giving up Einstein's definition of length contraction, Minkowski's calibration hyperbola (Passive LT) and Born's acceleration hyperbola (Active LT) become identical.

In §4 we introduce a *basic minimal acceleration* (Milgrom) in Lorentz boost with fundamental emission of radiation (CBR). By renormalization of Minkowski's metric, we show that classical "vacuum" is perfectly compatible with an expanding universe (Bondi's factor  $k$ ) where the Hubble radius  $a = R_H$  is an hyperbolic horizon. CR is distinct from the "Doubly Special Relativity" (DSR) approach, in that it does not look for a new invariance group adapted to Einstein's definition of distance, but for a new definition of distance, adapted to the LT.

**FIGURES** We suggest a geometrical version of this paper with **5 figures**. The principle of correspondence circle-hyperbola (PCCH) is deeply inscribed in Hyperbolic rotation (LT). This principle is valid for physical uniformly accelerated *circular* motion  $a = \frac{v^2}{r}$  and unfformly accelerated *hyperbolic* motion  $\alpha = \frac{c^2}{X}$ . Both motion have a center but if the first one is centripetal, the second one is **CENTRIFUGAL**.

## 1 Cosmological constant as second invariant in CSR (or CR)

Let us consider fundamental hyperbolas along  $Ot$  and  $Ox$  in Minkowski's space-time (at one space dimension with the axis,  $x$ ,  $t$  of system K with light velocity  $c = 1$ ). The two hyperbolas determining the units of measure ( $x^2 - t^2 = \pm 1$ ) are called hyperbolas of scale or calibration ([H. Minkowski 1908]). We focus the attention on the along  $Ox$  hyperbola defined with invariance of *space interval* by a *passive* Lorentz Transformation (*PLT*) (5)  $x' = OP' = O'P'$  ( $t' = 0$ ) (**Fig1**, light asymptotes and standard representation of primed axis  $x'$ ,  $t'$  "in scissors" or *hyperbolic rotation* of system K')

$$x^2 - t^2 = x'^2 \quad (\text{calibration hyperbola}) \qquad x^2 - t^2 = 0 \quad (\text{standard metric asymptotes}) \qquad (1)$$

Given that  $x'$  (the hyperbolic radius of curvature) can be as large as we wish, the calibration hyperbolas disappear at the infinity and we have only one invariant  $c = 1$  in standard configuration. Let us however note that, only the *finite* interval involves, according to Minkowski, that "space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality". Indeed an "infinite" interval ( $x^2 - t^2 = \infty$ , see 21) should mean that the independent space is given for any  $t$  and therefore the return of the shadow (Absolute Space  $Ox$ ,  $\forall t$ ). In order to stay in relativistic configuration suppose now that we have a very small but non nul constant  $\Lambda \neq 0$  in such a way that

$$R_C^2 = x'^2 = x^2 - t^2 \rightarrow \infty \quad x^2 - t^2 = \Lambda^{-1} \quad c = 1 \quad (2)$$

Such a reformed Minkowski's metric no longer is flat because we have an hyperbolic global curvature  $\varrho_h = \frac{1}{R_c} (\Lambda = \frac{1}{R_c^2})$ . In this case we have a constant radius  $R_C$  of the universe and therefore a static universe incompatible with the observed universe "in dilation". However, by the same argument with the other hyperbola ( $Ot$ ), we could claim that we should have an observed dilated time (in K, **Fig1**) of Universe (see §3.4).

Let us now underline that the right branch  $x > 0$  of  $Ox$  hyperbola can also represent a *worldline* of an uniformly ( $\alpha$ ) accelerated particle  $P'$ . According to Rindler (**Fig2**, [W. Rindler (1966)])

Consider a rod of arbitrary length resting along the  $x$  axis of Minkowski space. A time  $t = 0$  we wish to give one point of the rod a certain constant proper acceleration and we want the rod as a whole to move rigidly, i.e. in such a way that the proper length of each its infinitesimal elements is preserved. It turns out that each point of the rod must then move with a different though also constant proper acceleration, the necessary acceleration increasing in the negative direction and become infinite at a well-defined point of the rod; the rod can evidently not to be extended beyond or even quite up to that point, since an infinite proper acceleration corresponds to motion at the speed of light. If we arrange things so that this cutoff point lies originally at the origin the equation of motion of the point originally at  $x' = X$  is (2) We take  $X$  as a convenient spatial coordinate on the rod.

$$x^2 - t^2 = X^2 = \alpha^{-2} \quad (X > 0) \quad x^2 - t^2 = 0 \quad (x > 0) \quad (3)$$

(end of quotation). Born's rigid motion suppose a set of hyperbolic worldlines of points belonging to a rigid rod  $x' = O'P' = X$ . This basic motion with constant interval, is induced by successive *ALT* (Active Lorentz transformation, §2.2) which boosts the particle fixed in  $K'$  (the rigid rod) to a larger velocity. Given that in standard configuration the proper acceleration  $\alpha$  can be as small as we wish  $\alpha \rightarrow 0$  and so the hyperbola disappears at the infinity<sup>1</sup>: it only remains the light cone that defines the standard Minkowski's metric with only one invariant. With  $a = \gamma^3 \alpha = 0$  (14), the geodesics are *straight lines* in *flat standard* Minkowski's metric. However in special relativity (§3-2) the acceleration is a *spacelike* four-vector whose norm must necessarily be larger than zero  $\alpha > 0$ . Suppose now the existence of a minimal acceleration  $\alpha_M$  (3) which corresponds to the relativistic inexistence of an infinite interval (2).

$$\text{(left side)} \quad \frac{1}{R_H^2} = \varrho^2 = \Lambda = \alpha_M^2 = H^2 \quad \text{(right side)} \quad (c = 1) \quad (4)$$

where  $R_H$  is the observable radius of Hubble and  $\alpha_M$  is the observable acceleration of Milgrom ([M. Milgrom]). We rediscover the empirical standard values of constants  $x^2 - c^2 t^2 = \frac{c^4}{\alpha^2} = R_H^2$ . ( $c = 3.10^8 m/s$ ,  $T_H \approx 3.10^{17} s$ ,  $\alpha_M = 10^{-9} m/s^2$  [M. Milgrom MOND]). The cosmological constant ( $\Lambda = \frac{1}{R_H^2}$ , standard value defined by static 1917 Einstein's model) is a second invariant in our new SR which connects two previously independent empirical quantities  $R_H = cT_H$  and  $\alpha_M$ . This new completed SR that we suggest to call "*Cosmological (special) Relativity*" (**CR**) introduces in physics a *scalar* field of acceleration  $\alpha_M$  that is non-gravitational because the fundamental acceleration depends on the distance and not on the square of the distance.

The constant  $\Lambda$  can be seen either from the left side or the right side of equations (4).

In the first case (left side) we have a geometrical interpretation: the geodesic is no longer a straight spacetime world line but an hyperbolic spacetime worldline that can be connected with hyperbolic trajectories of

<sup>1</sup>We underline that in cosmology Minkowski's metric is characterized by the disappearance of Minkowski's calibration hyperbolas. Only the asymptotes survive to the extinction (only one invariant). So Minkowski's metric is a non-calibrated metric (see equation 21 renormalized by 34).

Pioneer sounds ([J. D. Anderson and all]). Exactly as in Euclidean geometry  $R^{-1}$  represents the global constant curvature of circle,  $\rho = R_H^{-1}$  represents in Minkowskian geometry the *global* constant curvature of the hyperbola. But the difference is that the radius  $r$  of the circle can be as great as we wish  $r \rightarrow \infty$  (globally the curvature is zero) the maximal interval-distance is  $R_H$  in CR (globally the curvature is non-zero) The basic heuristic in CR is based on the relativistic transformation of circle into hyperbola (that is well known, **Fig1**) and therefore also on the relativistic transformation of non-relativistic circular uniformly accelerated *motion* into hyperbolic rectilinear uniformly accelerated motion (that is not very well known, **Fig3**). This "principle of correspondence" "circle-hyperbola" (**PCCH**) plays a basic role in our analysis (5 Figures).

In the second case (right side) we have not only a cosmological interpretation of Milgrom's acceleration  $\Lambda = \alpha_M^2$  ([L. Blanchet]) but also, very curiously, Hubble physical constant  $H$  ( $c = 1$ ) =  $\alpha_M$ . The presence of this constant of expansion in (4) and therefore in (3 & 2) is very strange because Minkowski's space-time is renowned incompatible with expanding universe. The fact that the Hubble radius is a constant ( $R_H = R_C$ ) is not in contradiction with the observed expansion of the universe because we will prove that this radius is an **hyperbolic horizon** which involves the law of Hubble (37).

However there is a serious problem in *CR* because our theory is a purely logical building based on the identification (Fig1 & Fig2) between Minkowski's hyperbola (*PLT* without explicit acceleration) and Born's hyperbola (*ALT*, with an acceleration). This identification first involves the coincidence with  $O$  and  $O'$ , center of rotation in both cases in CR (§3) does not correspond to the most sophisticated model of rigid motion (§2, according to Rindler the "end ( $O'$ ) of rod is a photon"). Rindler's representation of Einstein's contraction involves that  $O'$  moves on the asymptote (its own infinite hyperbola) with respect to  $O$ . This is the reason why the two hyperbolas are generally not identified. Our identification involves a physical synthesis, based on **hyperbolic rotation motion**, between *PLT* and *ALT* and therefore the determination of a **fundamental acceleration in Lorentz boost** (§4).

## 2 Einstein-Born-Rindler's rigid motion and Cosmology

Einstein introduced his theory of contraction of length in 1905 (second paragraph) by using rigid rods ([A. Einstein (1905)]). Born defined in 1909 the *rigid motion* of each points of Einstein's rigid rod, by using accelerated hyperbolic motion of a particle ([M. Born]). Rindler found in 1966 an original metric in cosmology on the basis of Einstein-Born's rigid motion in special relativity ([W. Rindler (GR)]).

### 2.1 Einstein's contraction of rigid rod by *PLT*

Einstein considers two systems (rigid rods),  $K$  and  $K'$ , in standard configuration. Lorentz Transformation (LT), at one space dimension is:

$$x' = \gamma(x - \beta t) \quad t' = \gamma(t - \beta x) \quad (a) \quad x = \gamma(x' + \beta t') \quad t = \gamma(t' + \beta x') \quad (b) \quad (5)$$

with instantaneous coincidence of  $O \equiv O'$  at  $t = t' = 0$  and space-time  $c = 1$  units " $c = 1$ ",  $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ . In Einstein's paragraph 2 ([A. Einstein (1905)]), the system  $K'(x' = O'P' = L)$  is in uniform translation with respect to the system  $K$  for *any*  $t$  ( $t < 0$  and  $t \geq 0$ ). Einstein uses a *passive LT* in order to define the length in  $K$  of a moving rigid rod  $O'P'$  of  $K'$  with the simultaneity of two events  $t = 0$  in  $K$  (the difference of space coordinate of two ends  $O'$  and  $P'$  of the rigid rod at the same time in  $K$ ). By passive using of first *LT* (5a), we find the coordinates in  $K$  of  $O'$  and  $P'$ , respectively for finite (6a) and differential length (6b):

$$\begin{aligned} t = 0 &\Rightarrow x = \gamma^{-1}x' & (x^2 = x'^2 - t'^2) & \text{global (a)} \\ dt = 0 &\Rightarrow dx = \gamma^{-1}dx' & (dx^2 = dx'^2 - dt'^2) & \text{local (b)} \end{aligned} \quad (6)$$

### 2.2 Born's rectilinear uniformly accelerated rigid motion by *ALT*

Let us summarize briefly Born's procedure. In SR, the acceleration is not an invariant ( $a'$  and  $a$ ) and only a *proper* constant acceleration in  $K'$   $\alpha_{\text{Born}} = a'$  can be defined at the condition to do " $v' = 0$ " in the components of acceleration four-vector  $A' : \gamma'(\gamma' a'_x + \frac{d\gamma'}{dt'} v', \gamma'^3 v' \cdot \frac{dv'}{dt'} c)$  (at one space dimension):

$$v' = \frac{dx'}{dt'} = 0 \quad \text{and} \quad \gamma'(v') = 1 \quad \Rightarrow \quad \alpha = a' = \frac{dv'}{dt'} = \gamma^3(v) \frac{dv}{dt} = \frac{d(v\gamma)}{dt} \quad (7)$$

The comoving system with the particle  $P'$  is thus at each instant a new system,  $K'$ , defined by successive  $ALT$  with " $v' = 0$  but  $dv' \neq 0$ " ( $dv' = \gamma^2 \cdot dv$ ) and with " $t' = 0$  but  $dt' \neq 0$ ". In each  $K'$  system, the equation of the  $O'x'$  axis is  $t' = 0$ . The origin  $O'$  of successive  $K'$  is generally located in  $P'$  (Born); we will adopt the standard configuration (Rindler), by locating the particle  $P'$  at a distance  $x' = X$  (Born-Rindler's notations, [W. Rindler (SR)]) from  $O'$  in  $K'$  at the coordinate  $x_0 = X$  in  $K$  ( $t = t' = 0$ ). The differential equation ( $c = 1$ ) is  $\alpha dt = d(\gamma\beta)$ . A first integration is carried out with  $\beta = 0$  at  $t = 0$ :  $\alpha t = \gamma\beta$  or:

$$\alpha dt = d(\gamma\beta) \quad \implies \quad \beta = \frac{dx}{dt} = \frac{\alpha t}{\sqrt{1 + \alpha^2 t^2}} \quad (8)$$

A second integration yields  $x = \frac{1}{\alpha} \sqrt{1 + \alpha^2 t^2}$  with  $x_0 = \frac{1}{\alpha}$ , and therefore the right branch of hyperbola:

$$x^2 - t^2 = x'^2 = \alpha^{-2} = X^2 \quad (X \geq 0) \quad (a) \quad x = \gamma X \quad (b) \quad t' = t - \beta x = 0 \quad (c) \quad (9)$$

The accelerated hyperbolic motion is resolved by successive  $ALT$  into an infinity of instantaneous inertial motions (**Fig2**). This is Born's definition of *rigid motion* of the rod: at each instant  $t' = 0$ , the entire rod (and the infinitesimal rigid rod  $dX = dx'$ ) is at rest in its proper system  $K'$  (in Born-Rindler's notations  $X = \alpha^{-1}$ ). It is then impossible to separate the motion of the particle  $P'$  from the motion of the system  $K'$  (the rigid rod  $O'P'$ ) that drags it along. In other words, it is impossible to separate successive  $ALT$  (succession of events) from successive boosts of systems  $K'$ .

EBR's rigid motion is a relativistic interpretation of *rectilinear non-relativistic accelerated* motion which involves that the rod  $O'P'$  is in motion with respect to the origin  $O$ . So in Rindler's diagram  $(x, t)$  the end of the rod  $O'$  for  $t > 0$  is in motion on its own infinite hyperbola (**Fig2**), i.e. the asymptote. We underline that EBR's rigid motion is not correlated (PCCH, §1) with a rotation because in any rotation (including hyperbolic rotation) we have a center or a fixed point ( $O' \equiv O$ ). This is the reason why Rindler adopts a basic equation which result from differentiating for constant  $t$ ,  $x dx = X dX$ , introducing Einstein's contraction in rigid motion.

$$x = \gamma X \quad (\text{"dilation" by } ALT) \quad x dx = \overset{X dX}{\longleftarrow} \text{????} \quad dx = \gamma^{-1} dX \quad (\text{contraction by } PLT) \quad (10)$$

$K'$ . The problem in Einstein-Born's theory is that the entire rod  $O'P'$  is defined at constant  $t' = 0$  (simultaneity) and Einstein's contraction supposes also "at constant  $t$ " (also simultaneity, Rindler's representation with  $O'P'$  **Fig2**). The quadratic form (6a) is not the quadratic form (9a): there is an asymmetry (**Fig1**). We underline that in Born's theory the infinitesimal "proper length"  $dX = dx'$  ( $dt' = 0$ ) appears "as an invariant" exactly as Minkowski's proper time  $dt' = d\tau$  ( $dx' = 0$ ) appears as an invariant (see note 3). But in standard SR this analogy must be limited because if the space interval defines a distance exactly as the time interval defines a duration, we obtain a dilation (9b) in both cases.

### 2.3 Rindler's time parameter on rigid rod and cosmological non-Minkowskian metric

Rindler gives in 1966 a cosmological interpretation of Born's theory. He focuses at first the attention on the fact that there is in each point of the rod a "a different though also constant proper acceleration" but also on a singularity because the rigid rod (the "proper length") "ends in a photon" (infinite acceleration  $\alpha = \infty$ ). In order to develop analogy with Schwarzschild metric, Rindler then introduces a parameter  $T$  for observers with clocks on the rod.

It can be seen that the proper acceleration of the point  $X$  of the rod is  $1/X$ . Hence an observer at  $X$  feels a constant gravitational field of intensity  $1/X$ . The observers on the rod can so synchronize their clocks that each sees all the other clocks neither gain nor lose relative to his own; each observer must simply speed-up his proper clock by a factor equal to the reciprocal of his coordinate  $X$ . Let  $T$  denote this new time.

With Rindler's time parameter  $T = \beta_H$ , the hyperbolic velocity and not the proper time  $\beta_H = \alpha\tau$ , one obtains a non-Minkowskian metric

$$ds^2 = dx^2 - dt^2 = dX^2 - X^2 dT^2 = dX^2 - X^2 d\beta_H^2 \quad (a) \quad T = \alpha\tau \quad \rightarrow \quad \text{signature } (1, -X^2) \quad (b) \quad (11)$$

Ellis and Williams underline that Rindler's metric  $(1, -X^2)$  is based on "the boost-invariance of the proper length of the rod" or "boost-invariance of flat space-time" ([G. Ellis & R. Williams]). The fact that the "boost-invariance" involves a non-Minkowskian space-time is not consistent with the fact that the boosts in question are... Lorentz boosts. We need therefore a complete theory of *ALT* or Lorentz boosts (*hyperbolic rotations*). Rindler finally does not use the parameter  $\tau$  because the proper time  $\tau = \beta_H X$ , in a state of acceleration of  $K'$  systems, is a different one in each point of the rod<sup>2</sup>

### 3 Theory of acceleration in Lorentz boost (Hyperbolic ROTATION Motion and synthesis PLT-ALT)

EBR refers to Einstein's contraction (*PLT*, his paragraph 2) but not directly to Einstein's boost (his paragraph 3). In other words EBR's demi-hyperbola (**Fig2**) is not concentrated on the transition from 0-velocity to  $\beta$ -velocity (*ALT*) (**Fig3**). This is a natural interpretation of Born's integral hyperbola which is deduced from a differential equation (8a)  $\alpha dt = d(\gamma\beta)$  (unlike Minkowski's hyperbola). We now focus the attention on the element of *arc of hyperbola*  $P'_0P'$  (**Fig3**), i.e. on Einstein's boost.

Einstein considers two systems K and K', at rest relatively with each other. He places a rigid rod in K, of length  $L = OP$ , and an identical rigid rod in K', of identical length  $L = O'P'$  when both systems are at rest. Einstein then "boosts" the K' system at  $t = t' = 0$ , to bring it to a cruise speed of  $\beta$ ; this calls for an acceleration. Einstein explicitly states (in 1907) that the eventual consequences of this acceleration  $\alpha_{BOOST}$  on the  $O'P'$  rod disappear in K' as soon as the velocity becomes uniform (a statement we do not question). Einstein however does not say anything about this acceleration and how to compute this acceleration  $\alpha$ . Let us now develop into *three successive phases* (**Fig3**) a theory of acceleration in Einstein's boost with *ALT* ( $0 \rightarrow \beta$ ).

PHASE 1 The rod first is at rest ( $v = \beta = 0$ ) for  $t < 0$

PHASE 2 Then in  $t = t' = 0$  the rod is submitted in each point to a constant acceleration  $\alpha$  (*arc of hyperbolic rotation*  $P'_0P'$ ).

The element of angle that subtends the element hyperbolic arc is hyperbolic velocity  $d\beta_H = \alpha d\tau$

PHASE 3 Finally at  $t = \beta\gamma X > 0$ , the rod reaches its cruise speed of  $\beta$  (*tangent* in  $P'$ ,  $M'$  is the intersection with asymptote).

The end of the rod, the particle  $P'$  (the end of the rod) is transported by an active LT from the point  $P'_0(0, X)$  to the point  $P'(\gamma X, \gamma\beta X)$  in K i.e.

$$x^2 - t^2 = \alpha^{-2} = X^2 \quad (t \geq 0) \quad P(0) \xrightarrow{\text{hyperbolic arc}} P(\beta) \quad \alpha_{\text{Boost}}(P') = \alpha_{\text{Born}} \quad (ALT = \alpha LT) \quad (12)$$

Given that, in hyperbolic motion the particle  $P'$  and its proper comoving system  $K'(O')$  are completely inseparable,  $\alpha$  is therefore the acceleration in any Einstein's boost  $0 \rightarrow \beta$ . Our new theory of Einstein's boost of the rod  $O'P'$  is exactly Born's hyperbolic motion of particle  $P'$  if and only if we admit that  $O'$  coincide with  $O$  at  $t = t' = 0$ , for any velocity:  $\forall\beta$  (this is not true with Rindler's representation of contraction, see the contrast **Fig1&2**).

#### 3.1 Hyperbolic Rotation motion with *centrifugal* (expanding) acceleration

Our theory of Einstein's boost suppose that *ALT* must be consistent with *PLT*. In order to obtain this fundamental coherence (**Fig1** & **Fig3**), we have to admit that  $O'(\equiv O)$  (with  $\alpha = \infty$ ) reaches instantaneously the  $\beta$  - (or  $\beta_H$ ) - velocity. With the parametric equations of hyperbola  $x = \gamma X$ ,  $t = \beta\gamma X$

$$\begin{aligned} P'(X) &\xrightarrow{\text{Hyperbolic rotation (ALT)}} P'(\gamma X) \quad (\text{a}) \\ O &\equiv O' \quad (\forall\beta, \forall t : t = t' = 0) \quad \text{Center of Rotation} \quad (\text{b}) \end{aligned} \quad (13)$$

So the acceleration is clearly translated in hyperbolic motion by an **elongation** (13a) or extension of the space itself from the fixed point  $O'(\equiv O)$  into  $\gamma O'P'$  in K: Everything happens as if there was a fundamental "**elastic**" elongation  $\gamma X$  of distance  $X$ . We suggest to call this motion "*expanding motion*".

<sup>2</sup>Let us however think about the following "Gedanken Experiment". Suppose a rigid rod  $O'P'$  in prerelativistic kinematics at rest in the system K. If we push in  $t' = 0$  the rod with a finite constant acceleration in  $O'$ , instantaneously  $P'$  is in moving with the same finite constant acceleration. The two ends  $O'$  and  $P'$  of the rods are instantaneously boosted at the velocity  $\beta$  and therefore at the same Rindler's time  $T_{O'} = T_{P'}$ . This instantaneity at distance however cannot be true in a completely relativistic kinematics.

Given that, in non-relativistic rectilinear accelerated motion, the velocity is intrinsically non-constant, we admit a fundamental contrast (at one space dimension!) between a relativistic *hyperbolic elastic* motion ( $\frac{v^2}{X}$ ) and a non-relativistic *circular* motion ( $\frac{v^2}{r}$ ) because we have in both cases (PCCH) a constant velocity (respectively  $v$  and  $c$ ). In both cases (**Fig3**) we have also an acceleration in the direction of the center ( $O' \equiv O$ ): the *center* of the circle (13b) with constant distance (Euclidean orthogonality acceleration-velocity, §3-2) or the *center* of hyperbola (13b) with constant interval (Minkowskian orthogonality with velocity). In both cases  $O'$  does NOT move (in phase 2) with respect to  $O$  (unlike EBR, **Fig2**).

But if the acceleration in Euclidean rotation motion and Hyperbolic Rotation motion have the same direction, they have NOT the same sense: the former is centripetal and the latter is **centrifugal** (**Fig3**). So the basic expansion is inscribed in LT. Indeed the Principle of correspondence "circle-hyperbola" (PCCH) involves that the physical hyperbolic motion, induced by *ALT*, must be **Hyperbolic rotation** (LT), in geometrical meaning.

Elastic motion is not compatible with Einstein's contraction in rigid motion and this is the reason why we have to give up Einstein's definition of distance. In order to do that we have to define a passive use of LT that leads to a dilation of distance (13a). We will prove that the best way to precise Born's "boost-invariance" of the distance is to bring invariance of *proper distance* into line with invariance of *proper time* (new symmetry in SR, §3-4).

### 3.2 Non-zero norm of space-like 4-vector acceleration

The essential physical data of the material point in pre-relativistic kinematics are given by the 3-vectors position  $\vec{r}$ , velocity  $\vec{v}$  and acceleration  $\vec{a}$ . In relativistic kinematics, these data are replaced by the space-temporal 4-vectors  $\vec{R}$ ,  $\vec{U}$  and  $\vec{A}$ . The importance of the transition from 3-vector  $\vec{a}$  (pre-relativistic, a Galilean invariant) to that of a 4-vector  $\vec{A}$  is completely underestimated. The essential characteristic of  $\vec{A}$  is that it is a spacelike 4-vector, orthogonal, in the Minkowskian meaning, to the velocity 4-vector  $\vec{U}'$ , which is time like:  $\vec{U}' * \vec{A}' = \vec{U} * \vec{A} = 0$ .

We thus have the norms  $\|\vec{U}'\| = \|\vec{U}'\| < 0$  and  $\|\vec{A}'\| = \|\vec{A}'\| > 0$ , that must be *different from 0 (14)*, the zero norm being characteristic of a light-type 4-vector. In the general case of three space dimensions, the components of  $\vec{A}'$ ,  $\gamma'(\gamma' \mathbf{a}' + \frac{d\gamma'}{dt'} \mathbf{v}', \gamma'^3 v' \cdot \frac{dv'}{dt'} c)$  describe the motion of a particle  $P'$  in  $K'$ , which is also a motion in  $K$ . Let us restrict the 3-space dimensions problem to that of a single space dimension, in which case the space-time "4-vectors" become space-time 2-vectors  $\vec{A}'(A'_x, A'_t) \xrightarrow{LT} \vec{A}(A_x, A_t)$ . If the particle  $P'$  is now a rest (fixed) in  $K'$  ( $v' = 0$ ) spacelike 4-vector becomes 2-vector acceleration ( $a = \frac{dv}{dt}$ ,  $\alpha = a' = \frac{dv'}{dt'}$ ) and invariant norm of acceleration becomes proper acceleration:

$$\vec{A}'(\alpha, 0) \xrightarrow{PLT} \vec{A}(\gamma\alpha, \gamma\beta\alpha) \quad \|\vec{A}'\| = \|\vec{A}\| = \alpha \quad \alpha_{Boost} \neq 0 = \gamma^3 a \quad (14)$$

( $a_x = \gamma^2 a + \beta a_t = \gamma^2 a + \gamma\beta^2 \alpha = \gamma\alpha \implies \alpha = \gamma^3 a$ ). In **Fig3** the direction of acceleration along  $O'x'$  must be permanently orthogonal (in Minkowski's meaning) to the direction of velocity. The norm  $\alpha$  of *spacelike*  $\vec{A}$ , the proper acceleration in any Lorentz boost, cannot be equaled to zero (definition of lightlike 4-vector).

### 3.3 Proper time on elastic rod and renormalization of Minkowskian metric

We now examine the phase of acceleration in order to deduce firstly in  $K$  the duration of acceleration  $\Delta t_{P'}$  and, from the elongated distance, the distance of acceleration  $\Delta x_{P'}$

$$\Delta t_{P'} = \beta\gamma X_{P'} \quad \Delta x_{P'} = \gamma X - X = X(\gamma - 1) \quad (15)$$

with kinematics form  $\alpha\Delta x_{P'} = \gamma - 1 = \frac{v^2}{2} + \dots$  (Taylor) which corresponds to *relativistic kinetic energy*  $mc^2(\gamma - 1)$  of the particle  $P'$ . We obtain a different duration of acceleration for any point  $X$  of the rod ( $0 \leq t \leq \beta\gamma X$ ) We now calculate, by a simple integration, this duration *in proper time* of the Lorentz boost because we have  $t' = 0$  but  $dt' \neq 0$  with  $dx' = 0$  and with (8) (in Born-Rindler's notation we have in our theory for  $Ox$  hyperbola:  $dX = 0$ , note 3)

$$\tau_{P'} = \int dt' = \int d\tau = \int_0^{t_{P'}} \sqrt{1 - \beta^2} dt = \int_0^{t_{P'}} \sqrt{1 - \frac{\alpha^2 t^2}{1 + \alpha^2 t^2}} dt = \frac{1}{\alpha} \operatorname{arcsinh} \alpha t_{P'} \quad (16)$$

The transition between 0 and  $\beta$  velocity will be shorter  $\tau_{P'}$  ( $\approx \frac{1}{\alpha} \ln 2\alpha t_{P'}$ ) in proper time in  $K'$  than in time of  $K$ :  $\tau_{P'} < t_{P'}$ . By using *hyperbolic velocity*

$$\beta_H = \ln \sqrt{\frac{1+\beta}{1-\beta}} = \alpha\tau_{P'} \quad (17)$$

(the well known parameter of rapidity) we rediscover the hyperbolic coordinates of  $Ox$  hyperbola

$$t_{P'} = \frac{1}{\alpha} \sinh \alpha\tau_{P'} = \frac{1}{\alpha} \sinh \beta_H = \beta\gamma X \quad x_{P'} = \frac{1}{\alpha} \cosh \alpha\tau_{P'} = \frac{1}{\alpha} \cosh \beta_H = \gamma X \quad (18)$$

The proper time  $\tau_{P'}$  in order to reach the cruise velocity is a different one in each point of the rod in the phase of acceleration (phase 2). We have therefore the joint between the second phase and the third phase, the second limit of integration (because  $t' = 0$  but  $dt' \neq 0$ ):

$$t' = t - \beta x = \tau - \beta_H X = 0 \quad (19)$$

We rediscover the instantaneity  $\tau_{O'} = 0$  for one end  $O'$  of the rod and the maximal proper time  $\tau_{P'}$  for the other end of the rod. We have the following coupling for any Hyperbolic Rotation (HR) or any LT

$$d\beta_H = d\tau = \gamma^2 d\beta \quad (\text{a}) \quad (\text{differential hyperbolic angle}) \quad \beta_H = \alpha\tau \quad (\text{b}) \quad (\text{integral hyperbolic angle}) \quad (20)$$

given that  $\beta_H = \int d\beta_H = \int \alpha d\tau = \alpha \int_0^{t_{P'}} \sqrt{1-\beta^2} dt = \alpha \int_0^{t_{P'}} \sqrt{1 - \frac{\alpha^2 t^2}{1+\alpha^2 t^2}} dt = \text{arcsinh } \alpha t_{P'} = \alpha\tau_{P'}$ . Unlike Rindler (11) we admit that the proper time, defined  $\tau$  by (19) is the fundamental parameter on the rod. Let us remark that in proper time basic equation is of first order whose solution is an exponential (33) at the condition that  $\alpha \neq 0$ . So with our parameter  $\tau$  we obtain naturally a Minkowskian metric (1, -1) in phase2.

$$\begin{aligned} dt^2 - dx^2 &= d\tau^2 - (dx' = 0)^2 = \alpha^{-2} d\beta_H^2 & \Leftrightarrow & \quad x^2 - t^2 = \alpha^{-2} \quad \text{signature } (1, -1) \\ \alpha &= 0 \implies d\beta_H = 0 \implies dt^2 - dx^2 = \infty \cdot 0 = \text{underdeterminate} \\ x^2 - t^2 &= \infty = x^2 \quad (\text{non relativistic}) \end{aligned} \quad (21)$$

However we know that there is an hidden boost  $\alpha$  behind the "arc of hyperbola" in Minkowski's metric with  $ALT$  (**Fig3**). This unexpected result is geometrically obvious because if you take a point at rest and you apply a series of  $\alpha LT$  between two limits of integration (basic equation 19) the successive points will be placed not a straight line but... on an arc of hyperbola. Let us note that in SSR the infinitesimal interval is underdeterminate if  $\alpha = 0$  (Minkowski's standard metric is a non-calibrated metric and need a course of treatment (34) of renormalization (note 1) !

### 3.4 Synthesis between PLT and ALT and coupling "Born's acceleration and Minkowski's distance"

We come back now to Minkowski's hyperbolas (beginning of this paper). We focus the attention on the demi-branch (**Fig1**) of hyperbola  $x^2 - t^2 = x'^2 = d^2$  ( $x > 0, t \geq 0$ ). The distance  $d$  is defined by the interval between two simultaneous events in  $K'$

$$\vec{R}'(d, 0) \xrightarrow{PLT} \vec{R}(\gamma d, \gamma\beta d) \quad \left\| \vec{R}' \right\| = \left\| \vec{R} \right\| = d \quad x = \gamma d \quad (22)$$

Obviously  $d$  in  $K$  is no longer a "distance", in the usual meaning, but an interval. However, if we consider that a difference of space coordinates  $x$  in  $K$  defines a distance, then such a distance  $x = \gamma d$  is defined between two *non-simultaneous events* in  $K$  (gap of simultaneity:  $\gamma\beta d$ ). This is very interesting for astrophysical and cosmological distances §3-5). It is clearly a non-Einsteinian definition of (the dilation of) distance. However this new definition is induced from a perfect *symmetry in PLT* or in Minkowski's hyperbolas when we replace "events at the same place" by "events at the same time" in  $K'$  and "duration" by "distance" in  $K - K'$ . The duration  $D$  between "two events not at the same place" or the distance  $d$  between "two events not at the same time" is dilated in  $K$  ([Y. Pierseaux (2009)]). Both  $\alpha LT$  and  $dLT$  induce a dilation of distance. Minkowski's and Born's arc of hyperbola (*if  $d = X$* ), *are completely identical and so the couple* ( $\alpha_{Born}, d_{Minkowski}$ )

$$X_{Born} = d_{Minkowski} \quad \Rightarrow \quad d = \alpha^{-1} \quad \alpha LT \equiv dLT \quad (23)$$

has to be a fundamental relativistic couple. Unlike prerelativistic kinematics, in which there is no fundamental relationship between  $\vec{\mathbf{a}}$  and  $\vec{\mathbf{r}}$ , there is a relativistic four-vectorial relationship ()

$$(d, 0)(\alpha, 0) = \alpha_{\text{Born}} d_{\text{Minkowski}} = 1 \quad \vec{A}' \cdot \vec{R}' = \vec{U}^2 = c^2 = 1 \quad (\text{b}) \quad (24)$$

We verify with  $\alpha t = \beta \gamma$  and  $\alpha x = \gamma$ :  $\vec{A}' \cdot \vec{R}' = (x, t)(\gamma \alpha, \gamma \beta \alpha) = \alpha^2 x^2 - \alpha^2 t^2 = \alpha^2 x^2 - \alpha^2 t^2 = 1$ . The factor onehalf  $\frac{1}{2}$  ( $ad = \frac{1}{2}v^2 \implies E = \frac{1}{2}mv^2$ ) disappears in a relativistic theory ( $ad = c^2 \implies E = mc^2$ ).

### 3.5 Emission of radiation, Poincaré's definition of distance $d$ and Doppler redshift

According to Rindler "the rod ends in a photon". But in Born-Rindler's theory a photon dispatched to chase the particle at  $t = 0$  from  $O$  never catch up with it because the acceleration remains constant. In our new theory of Einstein's boost, the acceleration stops in the third phase and there is an intersection  $M'$  between the tangent and the worldline of the photon. In CR, Einstein's boost is structurally an emitter of radiation and this emission is not independent of the definition of distance.

#### 3.5.1 Synchronized clocks and length $L$ (Einstein 1905): rigid system and rigid rod

In the  $K'$ inertial system, let us first consider Einstein's rigid rod  $O'M'$  at rest in  $K$ (of length  $L$ ) with a source at  $O'$  and a mirror at  $M'$ . A light signal is emitted from  $O'$  at  $t' = 0$ , it is reflected in  $t' = \tau/2$  at  $M'$  and returns to  $O'$  at  $t' = \tau = 2L/c$ , the "time out"  $\frac{\tau}{2}$  being equal to the "back time"  $\frac{\tau}{2}$ . Einstein's clock synchronization uses three successive physical events, 1, 2, 3:  $O'(0, 0)_1$ ,  $M'(L, \frac{\tau}{2})_2$ ,  $O'(0, \tau)_3$ . He has synchronized the two clocks at the ends of the rod and defined the simultaneity of two events "at a distance". These two simultaneous events  $(0, \frac{\tau}{2})_2$  and  $(L, \frac{\tau}{2})_2$ , respectively at  $O'(x'_{O'} = 0)$  and  $M'(x'_{M'} = L)$  are however not explicitly given in Einstein's paper ( $\tau$  represent here Minkowski's proper time).

#### 3.5.2 Synchronous distance $d$ (Poincaré 1908): abstract systems and light-distance

The same system is used, but without Einstein's rigid rod (given a priori) and with a single clock in  $O'$ . We will suppose that the mirror  $M'$  is at rest in  $K'$  (a distant reflecting object) at an unknown distance. A light signal is emitted at  $O'$  at  $t' = 0$ , reflected in  $M'$  in  $D$  (Duration) and returns to  $O'$  at  $t' = 2D$ . The proper, "synchronous" distance  $d$  is defined by two simultaneous events in  $K'$ :

$$\text{simultaneity: } (0, D)_2 \text{ and } (d, D)_2 \quad (d = D)_{\text{Poincaré}} = \frac{1}{2} \tau_{\text{Minkowski}} \text{ (round trip)} \quad (25)$$

The "synchronous" distance  $d$  may be measured by a single clock in  $O'$  ( $2D$ ) with a round trip signal. Such a distance, measured by the light time of travel, is not a new technique. We can replace Einstein's rigid rod by "one half light travel time (*two ways or round trip*  $2D$ ) distance" with *Bondi's radar method* ([H. Bondi]). We add a new element: the converse (Poincaré) interpretation of Einstein's synchronization involves also that the so formed distance  $d$  is transformed by LT  $\gamma d$  like the duration  $\gamma D$ . In other words,  $d$  is a distance that is defined as the proper duration, and is thus the **shortest** in  $K'$ . As a last analysis and from a historical viewpoint, Einstein's work is based on the direct theorem (the  $O't'$  axis), while Poincaré's opened the way to the reciprocal (the  $O'x'$  axis). "This Lorentz hypothesis is the immediate translation of Michelson's experiment, if the lengths are defined by the time that light takes to travel through them" ([H. Poincaré 1908], [Y. Piereaux 2008] & [Y. Piereaux (2009)]). In the same way that "simultaneity at a distance" cannot be defined independently from the velocity of light, the distance itself may not be defined independently from this velocity. This *proper* distance  $d$  (events at the same time  $t' = 0$  in  $K'$ ) then becomes an "invariant" of LT ( $x^2 - t^2 = d^2 = s^2$ ) in the same sense as the proper duration  $D$  (events at the same place  $x = 0$  in  $K'$ ) is an "invariant" of LT ( $t^2 - x^2 = \tau^2 = s^2$ ). In summary Poincaré's proper length (hyperbola along  $Ox$ ) is the exact symmetric of Minkowski's proper time (hyperbola along  $Ot$ ); except the one-half  $\frac{1}{2}$  factor (twoways) in (25).

#### 3.5.3 One way light travel distance and Bondi's Doppler redshift factor (phase 3)

We remark now that the proportionality duration-distance, according to Poincaré's basic elongated ellipse [H. Poincaré 1908] is also valid for "one way" trip (without factor  $\frac{1}{2}$ ). We have in  $K$  because when the phase of acceleration stops

(phase 3), O' is in motion with respect to O:

$$\left(\sqrt{\frac{1-\beta}{1+\beta}}\right)_{forth} + \left(\sqrt{\frac{1+\beta}{1-\beta}}\right)_{back} = 2\gamma \quad (\text{round trip}) \quad d_{bzc} = kd(= D) \quad (26)$$

So the factor defining the distance by the travel time back is no longer  $\gamma$  but famous Bondi's factor  $k = \sqrt{\frac{1+\beta}{1-\beta}}$ . Both factors are Lorentz factor, the former in non-hyperbolic form of LT (5) and the latter in hyperbolic form of LT ( $\beta_H = \ln k$ ) or HR (remind  $x' = d$  and  $t' = D$  :

$$x = d \cosh \ln k \quad t = D \sinh \ln k \quad (\text{a}) \quad \beta = \tanh \ln k = \frac{k^2 - 1}{k^2 + 1} \quad \left(\frac{t}{x} = \beta\right) \quad (\text{b}) \quad (5\text{bis})$$

However in Bondi's theory, this factor is introduced in the transformation of the lengthwave and not the transformation of length itself: it is a Doppler factor. Given that, any one-way length is determined by the one-way light travel time, the transformation of *length* and *wavelength* are the same. In order to define the distance we must have a source in P'. Suppose a monochromatic source of lengthwave  $\lambda'$  in  $P'(K)$ . Observed at O in  $K$  we have the lengthwave  $\lambda$

$$\frac{d_{bzc}(\text{reception})}{d(\text{emission})} = \sqrt{\frac{1+\beta}{1-\beta}} = \frac{\lambda(\text{reception})}{\lambda'(\text{emission})} = k \quad (27)$$

We rediscover with Poincaré's dilation of distance the same Doppler formula as Einstein's one (longitudinally, the direct calculus with LT is very easy, [Y. Piereaux (2009)]). But with the theory of Lorentz boost we have structurally a redshift ( $\beta_H = \ln k$  and not  $\beta_H = \ln k^{-1}$ ). We suppose here that the acceleration stops (phase 3) and so we have (the derivation with respect of the proper time  $\tau$  is primed)

$$\alpha = \beta'_H = \frac{k'}{k} = 0 \quad \Rightarrow k(\tau) = cte \quad (28)$$

So in the framework of standard SR Bondi's factor is not an expansion factor (a scale factor) because it does not depend of proper time (see CR). Finally, we remark also that the one-way outgoing signal is still not used until now.

## 4 Fundamental Lorentz boost, minimal acceleration and one-way maximal distance in CR

Now we are able to transform SR into CR (§1). Until now we have a fundamental relationship between acceleration in Lorentz boost ( $\alpha LT$ ) and distance ( $dLT$ ) defined by interval in Minkowski's space-time (23). The cause of this acceleration is not inscribed in the structure of space-time and when the cause (the force) no longer acts (M'), the acceleration stops  $\alpha = 0 = \Lambda$  and we rediscover the standard SR with uniform cruise velocity or the straight line geodesic. In vacuum flat space-time we cannot place "on the same plane"  $(x, t)$  the  $ALT$  and the  $PLT$ . Without a fundamental acceleration  $\alpha_{\min}$  there is no Lorentz boost, no active LT, no hyperbola, and finally no complete SR. Without hyperbola, it remains the light cone, with only one invariant in standard SR. If we wish take into account the hyperbola, we have to consider the existence of a minimal ([M. Milgrom]) acceleration in the "vacuum".

$$a_{\min} \neq 0 = a_{boost} \Rightarrow a_{\min} d_{horizon} = a_M R_H = c^2 = 1 \quad \Leftrightarrow \quad x^2 - t^2 \not\rightarrow \infty \quad (29)$$

The existence of  $a_M$  (24) involves the existence of a fundamental Lorentz boost structurally connected with an emission of radiation. Given that distance  $d$  is measured in lightyear, if we send a light signal towards a galaxy we cannot wait for the return of the light. Our new definition of *proper distance*  $d$  supposes the existence of a "one way" (*without one half*)  $t' = D_{\text{one-way}}$  light travel time" distance, i.e. a *non-infinite including unit* in hyperbola (2). The radius of Hubble  $R_H$  is precisely defined by the one-way travel lifetime  $T_{\text{one-way}}$  of the universe or time of Hubble  $T_H$  (emission *CBR* cosmological background radiation, or photons structurally boosted fifteen billions

years ago (*fundamental acceleration involves fundamental emission of CBR*). With Poincaré's distance (25) we have: ( $t = 0$ , emission)

$$\text{simultaneity: } (0, T_H)_2 \text{ and } (R_H, T_H)_2 \quad R_H = T_H \text{ (one way)} \quad (30)$$

Bondi's mirror  $M'$  becomes Penzias-Wilson antenna, **Fig5**.

. So with the basic including hyperbolic unit we have now

$$d_{R_H} = r < R_H \quad \text{or} \quad r = \varepsilon R_H \quad 0 \leq \varepsilon \leq 1 \quad (31)$$

We will show that  $\varepsilon = \beta$ . On the other side, the introduction of a minimal acceleration  $a_M$  involves the deletion of the third phase of cruise velocity: any body in the vacuum undergoes a basic  $a_M - HR$ . The sound Pioneer  $P'$  follows in curved space-time a **geodesic** which is a generalization of inertial principle

#### 4.1 Milgrom's acceleration and cosmological acceleration of expansion

What is the relationship between Milgrom acceleration  $a_M = H$  ( $c = 1$ ) and the parameter of acceleration  $q_0$  in Cosmology? In other words what is the relationship with our constant Hubble radius  $R_H$  and the "ad hoc" scale factor in Friedman's metric (Robertson-Walker's metric), generally noted  $a(t)$  (with derivatives with respect to time,  $\dot{a}(t)$ ,  $\ddot{a}(t)$  and with  $q_0 = -\frac{\ddot{a}}{\dot{a}^2}$ )? In CR this factor is naturally equaled to  $q_0 = -1$  because we have  $\frac{a_M R_H}{c^2} = 1$ . So we have then an accelerated expansion in SR. For the remote galaxies  $r \lesssim R_H$  we obtain the same result with the law of expansion in  $k$  (factor of Bondi)  $(1+z)_{\text{Friedman}} = 1 + \beta + (1 + \frac{q_0}{2})\beta^2 + \dots = k_{\text{CR}} = \sqrt{\frac{1+\beta}{1-\beta}} = 1 + \beta + \frac{1}{2}\beta^2 + \dots$ . So we have:

$$a_M = H(c = 1) \quad \implies \quad q_0 = -1 \quad (32)$$

Unlike in Robertson-Walker's metric, we does not need an ad hoc factor in CR because Hubble constant  $H$  is now explicitly connected with the logarithmic derivation of expanding Bondi factor  $k$  in proper time (primed). Indeed fundamental equation is in CR is  $H = \frac{d\beta_H}{d\tau}$  and so the contrast

$$\begin{aligned} H &= \frac{\dot{a}(t)}{a(t)} \text{ (Friedman-Lemaître)} & H &= \beta'_H = \frac{k'(\tau)}{k(\tau)} = \alpha_M \text{ (CR)} \\ \implies & k(\tau) = k_0 e^{H\tau} = e^{H\tau} \end{aligned} \quad (33)$$

because  $k(0) = 1$  in the boost. And so the non nul Hubble constant involves an exponential expansion compatible with hyperbolic constant horizon. In summary we have switched to the second member  $dt^2 - dx^2 = a(t) = Cte$  Friedman's scale factor ( $dt^2 - a(t)dx^2$ ) by transforming it into constant. Then we showed that the scale factor of expansion is Bondi's factor  $k$  and we have therefore to discover a relationship between  $R_H$  and  $k$  (36).

#### 4.2 Renormalized Minkowski's metric, hyperbolic horizon and global Hubble's law

The Lorentz boost describes the continuous transformation from any point of the elastic rod from the velocity 0 to  $\beta$ . The element of proper time  $d\tau$  depends in CR on Hubble constant: With  $\alpha_{Min}$  we have a renormalized Minkowski's metric (21) by introducing a fundamental unit  $R_H$

$$dt^2 - dx^2 = d\tau^2 = H^{-2}d\beta_H^2 = R_H^2 d\beta_H^2 \quad \frac{d\beta_H}{d\tau} = \frac{k'}{k} \quad \text{(LOCAL)} \quad (34)$$

Locally for  $r \ll R_H$  ( $\alpha \gg \alpha_{Min}$ ) standard flat metric of SSR is still valid. Indeed locally the hyperbola becomes a parabola  $x = \frac{c^2}{a}(1 + \frac{\alpha^2 t^2}{2c^2} + \dots) = \frac{c^2}{a} + \frac{1}{2}\alpha_0 t^2 + \dots$  and parabolic Einstein's principle of equivalence on the basis of GR is still valid but *globally* we have the curvature of fundamental Milgrom's (demi-branch of) hyperbola  $\alpha = \varrho$  (**Fig5**).

$$x^2 - t^2 = R_H^2 \quad \beta_H = H\tau = \ln k \quad \text{(GLOBAL)} \quad (35)$$

Let us finally show that the geometry is hyperbolic in the meaning of Cayley and Klein.. With Cayley-Klein's hyperbolic distance  $r_H$  induced from the *cross-ratio* formula is given by  $r_H = r_h \operatorname{arctanh} \frac{r}{r_h} = \ln \sqrt{\frac{1+\frac{r}{r_h}}{1-\frac{r}{r_h}}}$  where  $r$ , is a Cartesian distance smaller than  $r_h$  the horizon. We have from the global form  $\tau_{P'} = \frac{1}{H} \beta_H = T_H \ln \sqrt{\frac{1+\beta}{1-\beta}}$  and therefore  $\tau_{P'} = T_H \ln \sqrt{\frac{1+\frac{\beta T_H}{t_h}}{1-\frac{\beta T_H}{t_h}}}$ . With  $\xi_{P'} = c \tau_{P'} (c = 1)$  and  $R_H = c T_H (c = 1)$  ( $\xi_{P'}$  is the proper distance<sup>3</sup> with element of proper distance  $d\xi_{P'} = dx'$ ), we have a basic relationship between  $k$  and  $R_H$

$$\tau = T_H \ln \sqrt{\frac{1 + \frac{\beta T_H}{t_h}}{1 - \frac{\beta T_H}{t_h}}} = \ln k \quad \Rightarrow \quad \xi = R_H \ln \sqrt{\frac{1 + \frac{\beta R_H}{R_H}}{1 - \frac{\beta R_H}{R_H}}} = R_H \ln \sqrt{\frac{1 + \frac{r}{R_H}}{1 - \frac{r}{R_H}}} \quad (36)$$

where  $k$  is the redshift Doppler factor of Bondi [H. Bondi ]. This is the inverse form of (33). And so we deduced, with  $r \lesssim R_H$  the law of Hubble ( $0 \leq \beta \leq 1$ )

$$\beta = Hr \quad (37)$$

From integral form , we can therefore deduce globally the law of Hubble. In CR, like in Esher's drawing, there is a constant hyperbolic horizon and an expanding Universe with potentially infinite hyperbolic distances, ( $x \rightarrow \infty$  or  $\xi \rightarrow \infty$ ) in time  $t$  and proper time  $\tau$  as well. The Universe in CR (33-36) is not static but in "steady state"[H. Bondi ]. Nothing is changed for a light point because we have obviously  $ds^2 = d\tau^2 = dt^2 - dx^2 = 0$ . Is the light sensitive to the curvature in CR or to cosmological constant  $\Lambda$  ([W. Rindler & M. Ishak (2007)]) Given that light follows in "vacuum" a straight line and that straight line is in CR hyperbolic straight line (with curvature), the answer is therefore positive.

## 5 Conclusion: Einstein's cosmological constant and Poincaré's negative pressure of the "vacuum"

Our main heuristic principle is the principle of correspondence non-relativistic and relativistic kinematics (PCCH). We have the theater of circular motion and the theater of hyperbolic rotation motion. The roles but not the actors are the same. Let us give an example. Who plays the role of angular velocity  $v = R\omega$ ? In uniform circular motion, unlike constant angular velocity  $\omega$  (in standard units  $s^{-1}$ ), linear velocity  $v$  depends on radius. In hyperbolic motion linear velocity depend on the radius  $c = R_H H$  unlike Hubble constant (in standard units  $s^{-1}$ ) which does not depend of distance (37)  $\beta = rH$ .

Last but not least: Who plays the role of the transcendent  $\pi$  in hyperbolic CR? The fundamental differential equation in CR, based on  $a_M - HR$ , is a first order equation (33-36). Thanks to  $\Lambda \neq 0$  the solution is not trivial and CR is exactly the completely hyperbolic theory of hyperbolic rotation, i.e. for Lorentz boost with Milgrom's acceleration. The circular motion governed by  $\pi$  is replaced in CR by an expanding hyperbolic motion governing by  $e$ . This complete substitution characterizes the aesthetic superiority of hyperbolic geometry on the other non-Euclidean geometries [R. Penrose].

Einstein had introduced In 1917 the cosmological constant  $\Lambda$  in the first member of Einstein's GR equation; if it is switched to the second member, as is now the general practice, it becomes a term independent from matter in the usual sense of the word. But what is the meaning of a constant, unrelated with matter, in a theory (GR) which by essence gives a structure based on matter to the space-time? We suggest, as a consequence, not to shift it from one member to the other, but to change its relativity in order to reinsert it into its natural "environment", in other words, in the "vacuum" space-time of Minkowski.

Nothing is changed with GR except that  $\Lambda$  must be introduced in Einstein's equation. The cosmological constant,  $\Lambda$ , when it is written on the right side, introduces the constant global curvature of the universe ([E. Gunzig]), as distinct from all the other terms of Einstein's GR equation, which correspond to the local curvature (defined by gravitation in each point of the space-time). The fact that the acceleration may not be null is not in contradiction with the local Riemann flatness of the pseudo-Euclidean space, as derived from the

<sup>3</sup>The element of proper distance  $d\xi$  is connected by the other hyperbola  $Ot$  (Fig1). We have a perfect symmetry between proper time  $d\tau = dt'$  and proper length  $d\xi = dx'$  with the other ( $Ot$ ) hyperbola  $t^2 - x^2 = T_H^2$  and  $d\xi^2 = dx^2 - dt^2$  if  $d\xi = id\tau$ .

affine connection (the "Christoffel"), under the condition that  $\Lambda$ , defined by CR, be introduced into Einstein's GR equation.

Our use of inverted commas in the title of our paper means that the classical "vacuum" is not really empty because there is a fundamental field that can be called "Poincaré's relativistic aether" or "hidden relativistic fluid" or in "dark energy" (it's the fashion). Given that nothing is changed between the relation with GR except that the cosmological constant is imposed by CR, we can determine thanks to Einstein's GR, and more precisely from Einstein's static model of Universe, the very weak density of Poincaré's fluid with Poincaré's *non-electromagnetic* negative pressure ([H. Poincaré 1908])

$$\rho_{\text{hidden fluid}} = \frac{\Lambda_{\text{Einstein}}}{4\pi G} = -p_{\text{Poincaré}} \quad (38)$$

If Einstein is the author of the very well known  $E = mc^2$ , Poincaré is the author of the very unknown  $p = -\rho c^2$  in the framework of LT. Einstein considered at the end of his life that the cosmological constant  $\Lambda$  in 1917 was his "greatest mistake". Maybe Poincaré thought also, at the end of his life, that his "greatest mistake" was keeping aether in his relativistic dynamics of 1906 (a relativistic aether but an aether)? According to our last formula, maybe both, the "old Poincaré" and the "old Einstein", were wrong. If we put Einstein's cosmological constant in his own relativistic kinematics without aether, we rediscover Poincaré's relativistic aether (La "structure fine" de la Relativité Restreinte, [Y. Pierseaux 1999]). This is the irony of the history.

## References

- [H. Minkowski 1908] "Raum und Zeit", Cologne, Phys. Z., 20, 1909, 104-111. Teubner, Leipzig - Berlin, 5ème édition, 1923, 54 - 71.
- [W. Rindler (1966)] "Kruskal space and the uniformly Accelerated frame", Am. J.of Physics, Vol. 34, Issue 12, December 1966, 1174-1178.
- [W. Rindler (SR) ] "Relativity. Special, General and cosmological", Oxford University Press, 2001.
- [W. Rindler (GR)] "Introduction to special relativity", Oxford Science Publication.
- [W. Rindler & M. Ishak (2007)] "The Contribution of the Cosmological Constant to the Relativistic Bending of Light Revisited", Phys. Rev. D 76, 043006 (2007) [5 pages]
- [M. Milgrom] "A Modification of the Newtonian Dynamics as a Possible Alternative to the Hidden Mass", Astrophys J., 270, 365-370, 1983.
- [M. Milgrom MOND] "The MOND paradigme", arXiv:08013133v2[astro-ph] 3 Mars 2008.
- [J. D. Anderson and all] "Study of the anomalous acceleration of Pioneer 10 and 11". Phys. Rev. D. 65, 082004 (2002)
- [L. Blanchet] "Dipolar Dark matter and MOND", séminaire à l'ULB, invité le 23 février 2009 par B. Famaey.
- [E. Gunzig] Que faisiez-vous avant le big bang? Odile Jacob, 2008, Paris.
- [A. Einstein (1905)] "Zur Elektrodynamik bewegter Körper", Ann.d.Ph,17, p892-921,1905.
- [M. Born ] "Die theorie des Starren Elekctrons in Kinematics des RelativitätPrinzip", Ann. Phys, Band 30 (1909), 1-56.
- [G. Ellis & R. Williams] "Flat and curved space-times", Oxford University Press, 2000.
- [H. Poincaré 1908] "La dynamique de l'électron". R. Gén. des Sciences Pures et Appl., 19, 386-402, 1908.
- [Y. Pierseaux 2008] "La cinématique sous-jacente à l'ellipse de Poincaré", C.R. Physique, n° 7-8, p921-929, 2007, Paris

- [Y. Pierseaux 2004] "Einstein's spherical light waves versus Poincaré's ellipsoidal light waves", An. F. de Broglie, vol 30, n°3 – 4, 2005.
- [Y. Pierseaux (2009)] "Du boost de Lorentz à l'accélération de Milgrom", IIHE, 2009-01, ULB-VUB, Pleinlaan, 2, 1050 Brussels.
- [Y. Pierseaux 1999] "La structure fine de la Relativité Restreinte, L'Harmattan, Paris, 1999.
- [H. Bondi ] "Assumption and Myth in Physical Theory", Cambridge University Press, 1967.
- [R. Penrose] "The large, the small and the human mind", Cambridge University press, 2000.