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USING RELATIVE VELOCITIES AND HYPERBOLIC GEOMETRY IN SPECIAL RELATIVITY

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Abstract

Although in principle special relativity is based on the use of relative motions, the Einstein formulation of 1905 did not discuss relative velocities directly. It was only many years later that Fock (1955) gave an analysis of relative velocity showing its relationship with hyperbolic (Bolyai-Lobachevski) geometry. Varićak (1912 etc.) and others had shown the importance of this geometry for the special theory but their fundamental contributions did not pass into the mainstream of physics. The Beltrami-Klein representation of hyperbolic space provides a very convenient geometrical illustration for relative velocity and the analogue of ordinary vector addition and subtraction can be used in geometrical diagrams for sequentially ordered relative velocities.

The relative velocity formula of Fock actually coincides with the addition formula given by Einstein modified to apply to velocity difference. It leads to the strange result that for two moving points P_1, P_2 the velocity of P_2 relative to P_1 is not the negative of the velocity of P_1 relative to P_2, a spatial rotation occurring. A similar phenomenon for velocity addition was discussed later by Mocanu (1986) and called 'the Mocanu paradox' by Ungar (1989) who proposed to explain it by Thomas rotation.

The present paper puts forward another view on this subject. It is pointed out that Sommerfeld in 1908 had already discussed similar effects with velocity addition associated with noncommutativity, contraction and rotation. These effects can be explained simply as arising from projection

from hyperbolic geometry into the observed Cartesian form. The Mocanu paradox must have a similar explanation.

The related difficulty discussed by Ungar of non-associativity and non-commutativity of velocity composition can be resolved by an approach to velocity composition briefly described by the writer (2004) based on using velocity four vectors to give matrix representation of relative velocity. For three moving points P_1, P_2, P_3, the relative velocity matrix of P_3 to P_1 is the product of the relative velocity matrices of P_3 to P_2 and P_2 to P_1 - a result which clearly extends to a chain of relative velocities. This property is independent of origin. In this approach only transformations corresponding to sequentially ordered relative motions may be combined so that, for example, the product of two pure Lorentz matrices ("boosts") usually has only mathematical meaning lacking physical interpretation.

Hyperbolic (Bolyai-Lobachevski) Geometry

The axiom of parallels of Euclid's geometry may be stated that the left asymptotic parallel at a point P to a given line coincides with the right asymptotic parallel – see the figure where the right asymptotic parallel is the limiting position of lines PQ, PQ', ... etc. and similarly on the left.

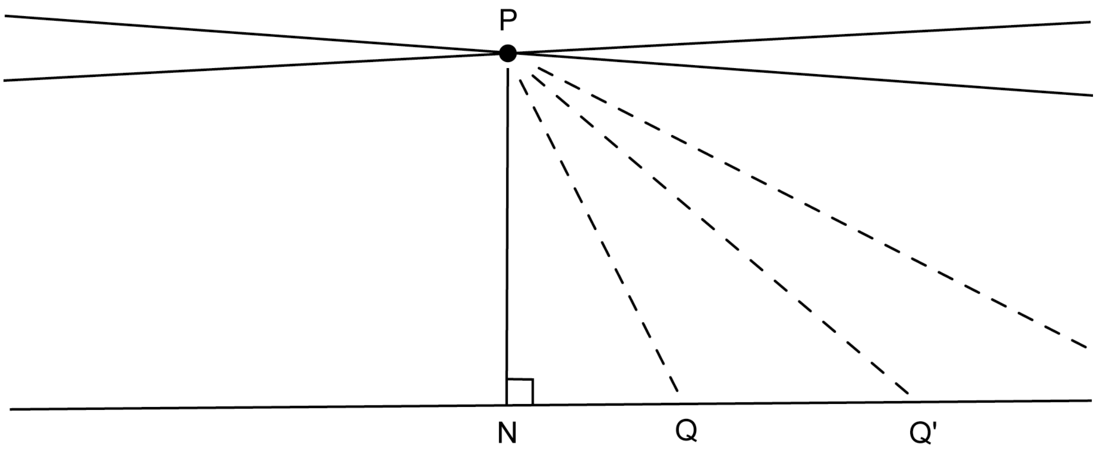


Fig: The axiom of parallels

The failure of this axiom leads to a logically consistent geometry - *hyperbolic geometry* also known as *Bolyai -Lobachevski geometry* after the two principal discoverers: Bolyai (1802-1860) and Lobachevski (1792-1856)

It is possible, even likely, that the geometry of our space is hyperbolic, the curvature being so small that the hyperbolic nature is only observable on a cosmological scale so that at terrestrial distances the difference from Euclidean space is beyond possible experimental observation. Thus the cosmology of Milne (equivalent to the asymptotic hyperbolic Friedmann model) indicates an angle between left- and right asymptotic parallels of the order of 10^{-26} degrees for a displacement of a few meters from a line.

Most importantly velocity space is hyperbolic and in this case the hyperbolic nature is observable by the phenomena of special relativity.

Two Approaches to Special Relativity

The Minkowski velocity 4-vector was originally defined using proper time τ as

$$(U_1, U_2, U_3, U_4) = (dx/d\tau, dy/d\tau, dz/d\tau, ic\ dt/d\tau)$$

The components satisfy identically

$$U_1^2+U_2^2+U_3^2 + U_4^2 = (ic)^2$$

They represent a point on a 4 dimensional sphere of imaginary radius ic. The rotations of this sphere are the Lorentz transformations. Under the influence of Minkowski this became the standard approach.

The change over to the hyperbolic theory consists in avoiding the use of the imaginary time coordinate and using only real coordinates. So the modification of the definition of Minkowski is that velocity four vector is defined as

$$\mathbf{U} = (U_0, U_1, U_2, U_3) = (c\ dt/d\tau, dx/d\tau, dy/d\tau, dz/d\tau)$$

Note that here U_1, U_2, U_3 are the same as before. The components now satisfy identically the equation

$$U_0^2 - (U_1^2+U_2^2+U_3^2) = c^2$$

This defines a hyperbolic surface invariant under Lorentz transformation. The geometry is now hyperbolic and this approach, followed consistently, leads to the hyperbolic theory of special relativity due principally to Vladimir Varićak (1865-1942), Professor of Mathematics at Zagreb University.

The Beltrami-Klein representation

An important representation of plane hyperbolic geometry is that of Beltrami-Klein. This is by the geometry of straight line segments within a circle. The circle is regarded as a circle at infinity or infinite horizon so that segments meeting on the circle are asymptotically parallel (figure).

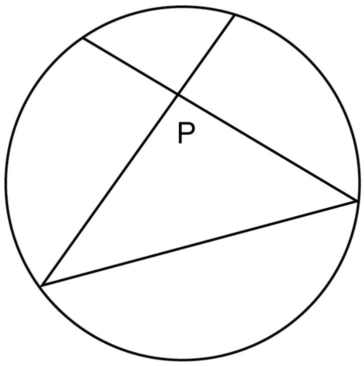


Fig. Asymptotic parallels from a point P

Beltrami was the first to discuss this representation which he did using a Riemannian metric. He showed that geodesics are straight lines and that there is an associated distance now known as the *Cayley-Klein metric*. The radial Cayley-Klein distance ρ from the centre point O to a point radial distance r is

$$\rho \ = \ R \ \mathrm{th}^{[-1]} (r/R)$$

As r increases from 0 to R, ρ goes from 0 to infinity and so the interior of the circle maps into an infinite space which has the Riemannian metric

$$d\rho^2 \ + \ R^2 \, \mathrm{sh}^2(\rho/R) \, d\theta^2$$

There is a similar mapping of the interior of a sphere into a 3 dimensional hyperbolic space with metric

$$d\rho^2 \ + \ R^2 \, \mathrm{sh}^2(\rho/R) \, [d\theta^2 \ + \sin^2\theta \, d\varphi^2]$$

Rapidity and Hyperbolic Velocity

Rapidity w is related to velocity v by

$$v = c\ \mathrm{th}\ w \qquad -\infty < w < \infty$$
$$w = \mathrm{th}^{[-1]}(v/c) \qquad -c < v < c$$

From these equations follow

$$\mathrm{ch}\ w = \frac{1}{\sqrt{(1-v^2/c^2)}} \qquad \mathrm{sh}\ w = \frac{(v/c)}{\sqrt{(1-v^2/c^2)}}$$

Rapidity has the important property of *additivity.* Suppose rapidities w_1, w_2, w correspond to velocities v_1, v_2, v. Then to the composition law

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}$$

corresponds

$$w = w_1 + w_2$$

This can be seen from the identity

$$\mathrm{th}\ (w_1+w_2) = \frac{\mathrm{th}\ w_1 + \mathrm{th}\ w_2}{1 + \mathrm{th}\ w_1\ \mathrm{th}\ w_2}$$

Hyperbolic velocity V is rapidity w scaled to approximate ordinary velocity v when v/c is small:

$$V = c\,w = c\ \mathrm{th}^{[-1]}(v/c)$$

It is additive like rapidity and is more appropriate for physics

Beltrami Velocity Space

Velocities are restricted by the inequality

$$v_x^2 + v_y^2 + v_z^2 < c^2$$

The velocity (kinematic) space is the interior of a sphere parametrized by spherical coordinates θ, φ as

$$v_x = v \sin\theta \cos\varphi$$
$$v_y = v \sin\theta \sin\varphi$$
$$v_z = v \cos\theta$$

With rapidity w (or hyperbolic velocity V) as radial distance

$$v_x = c \,\text{th}\, w \sin\theta \cos\varphi = c \,\text{th}\, (V/c) \sin\theta \cos\varphi$$
$$v_y = c \,\text{th}\, w \sin\theta \sin\varphi = c \,\text{th}\, (V/c) \sin\theta \sin\varphi$$
$$v_z = c \,\text{th}\, w \cos\theta = c \,\text{th}\, (V/c) \cos\theta$$

It is now an infinite hyperbolic space

Formulae for relative velocity: Suppose two moving points P, Q have rapidities w_1, w_2 relative to origin O.

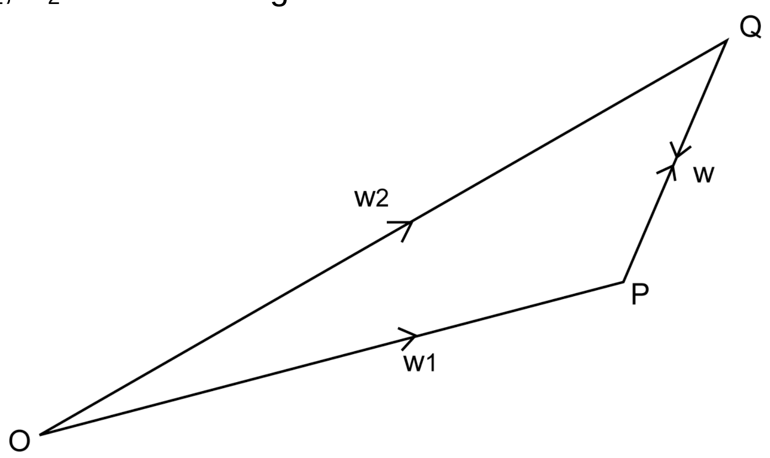


Fig: Combination of rapidities

Forming a triangle with rapidities as shown the third side has rapidity w given by the hyperbolic cosine rule for angle POQ (= θ)

$$\text{ch}\, w = \text{ch}\, w_1 \,\text{ch}\, w_2 - \text{sh}\, w_1 \,\text{sh}\, w_2 \cos\theta$$

or in terms of hyperbolic velocity V

$$\text{ch}\, V/c = \text{ch}\, V_1/c \,\text{ch}\, V_2/c - \text{sh}\, V_1/c \,\text{sh}\, V_2/c \cos\theta$$

On using the previous identities relating rapidities w and velocities v: follows

$$\text{ch } w = \frac{1}{\sqrt{(1- v^2/c^2)}} = \frac{1 - (v_1/c)(v_2/c) \cos\theta}{\sqrt{(1 - v_1^2/c^2)}\sqrt{(1 - v_2^2/c^2)}}$$

Solving for v^2 gives Einstein's formula for velocity difference:

$$v^2 = \frac{\{v_1^2 + v_2^2 - 2 v_1 v_2 \cos\theta - (v_1 v_2/c \sin\theta)^2\}}{(1 - v_1.v_2 /c^2 \cos\theta)^2}$$

Consequently w is the relative rapidity of P and Q equal to both $w_{2/1}$ and $w_{1/2}$ in magnitude.

The Cayley-Klein metric: The quantity w gives a metric for separation of two rapidities. The abive formula for ch w may be written as

$$\text{ch } w = \frac{c^2 - \mathbf{v}_1.\mathbf{v}_2}{\sqrt{(c^2 - \mathbf{v}_1.\mathbf{v}_1)}\sqrt{(c^2 - \mathbf{v}_2.\mathbf{v}_2)}}.$$

From this comes the Cayley-Klein metric:

$$w = \text{ch}^{[-1]} \frac{c^2 - \mathbf{v}_1.\mathbf{v}_2}{\sqrt{(c^2 - \mathbf{v}_1.\mathbf{v}_1)}\sqrt{(c^2 - \mathbf{v}_2.\mathbf{v}_2)}}.$$

The Riemannian metric: This can be introduced by considering the magnitude squared of the differential relative velocity of points having velocities **v, v** + d**v** resulting in the polar coordinate expression

$$c^2\, dw^2 + c^2\, \text{sh}^2 w\, [d\varphi^2 + \sin^2\varphi\, d\theta^2]$$

In terms of the corresponding hyperbolic velocity V it is

$$dV^2 + c^2\, \text{sh}^2 V/c\, [d\varphi^2 + \sin^2\varphi\, d\theta^2]$$

This is the standard form for Riemannian metric of a hyperbolic space of radius of negative curvature c

Addition of relative velocities: Beltrami space is convenient because velocities are represented by straight lines (even though angles are not preserved). This is illustrated below for three moving points B, C, D having hyperbolic velocities V_1, V_2, V_3 relative to origin O. The relative velocities are found by completing triangles as shown giving the triangular composition of two relative hyperbolic velocities $V_{2/1}$ and $V_{3/2}$ to give $V_{3/1}$

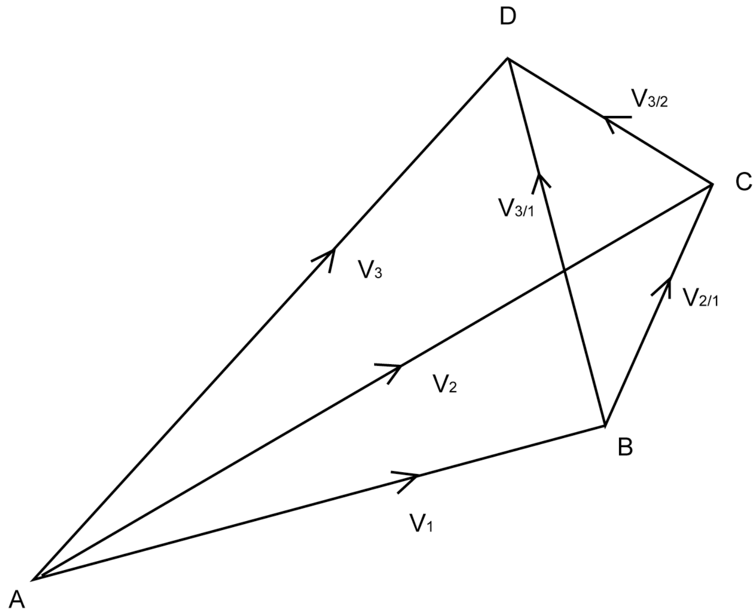


Fig: Combination of relative velocities

This figure corresponds to the matrix composition rule for relative velocities considered later.

Non-commutativity of velocity composition

The combination of the velocities $(v_1, 0)$, $(0, v_2)$ in the two possible ways (the first followed by the second or vice versa) results in corresponding velocities

$$(v_1,\ v_2 \sqrt{(1 - v_1^2/c^2)}),\quad (v_1\sqrt{(1 - v_2^2/c^2)},\ v_2)$$

They have the same squared magnitude:

$$v^2 = v_1^2 + v_2^2\,(1 - v_1^2/c^2) = v_1^2\,(1 - v_2^2/c^2) + v_2^2$$

From them are found Pythagoras formulae leading to the figure below where resultant velocities are represented by lines AC and C'A' of equal length.

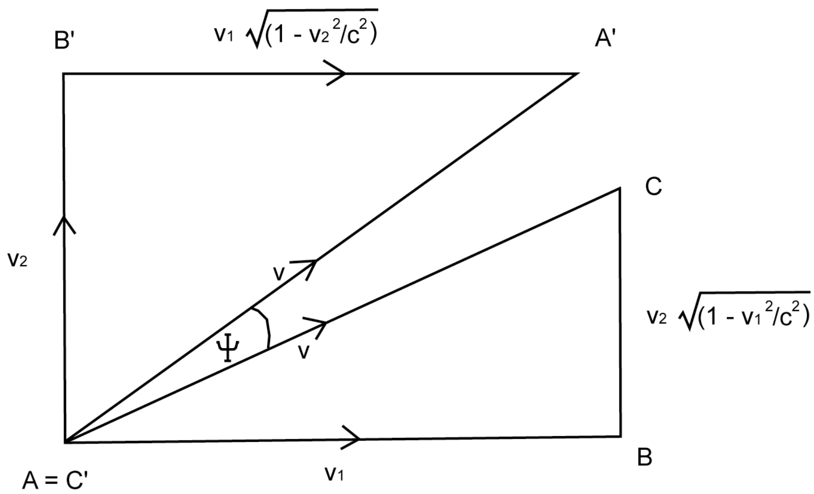


Fig: Sommerfeld's figure showing non-commutativity of orthogonal velocity composition (Cartesian form)

Multiplication of the transverse velocity components by contraction factors results in a failure of the rectangular figure to close. This was first observed by Sommerfeld (1908) who called it *non-commutativity of velocity addition*. Trigonometric functions of angle Ψ between the resultants are easily found from this diagram by taking scalar and vector products of vectors AC, C'A'.

In this simple case it is easy to show that Lorentz matrix multiplication does not lead to correct velocity composition. Thus v_1 horizontally and v_2 vertically would give resultant v, not along AC, but along C'A' Similarly v_2 vertically and v_1 horizontally would give resultant v along AC.

Projection and the Contraction Factor

In the hyperbolic theory the physical state exists in hyperbolic space. This is projected on to our Cartesian perception of it in a similar way to the projection of a spherical surface on to a tangential plane. We are unaware of the curvature. Projection gives rise to the contraction factor characteristic of relativity formulae, e.g. for a right-angled triangle:

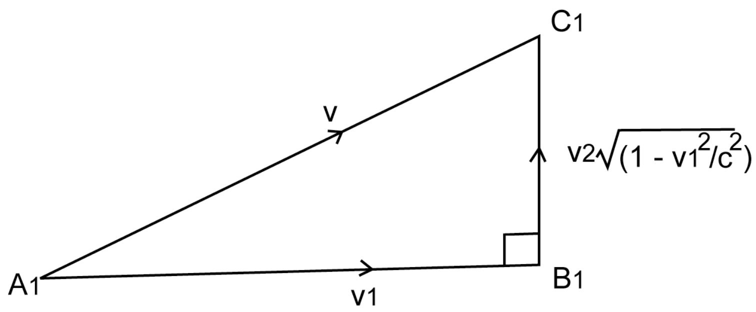


Fig: A right angled velocity triangle in Cartesian form

The corresponding figure in hyperbolic space would be

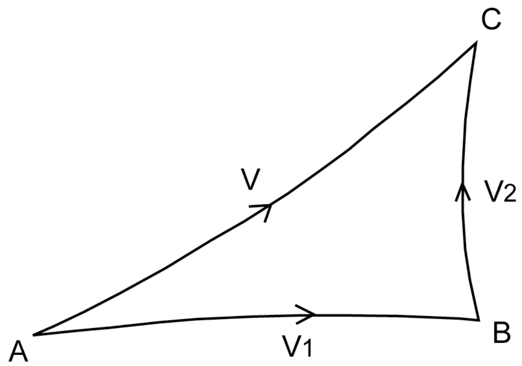


Fig: The right angled triangle in hyperbolic form

This diagram is of course only schematic because exact representation is not possible on a Euclidean plane.

Sommerfeld's Figure in Hyperbolic space

The last two diagrams explain the appearance of Sommerfeld's figure: Cartesian projection loses the symmetry of the corresponding hyperbolic figure which would be as indicated schematically below:

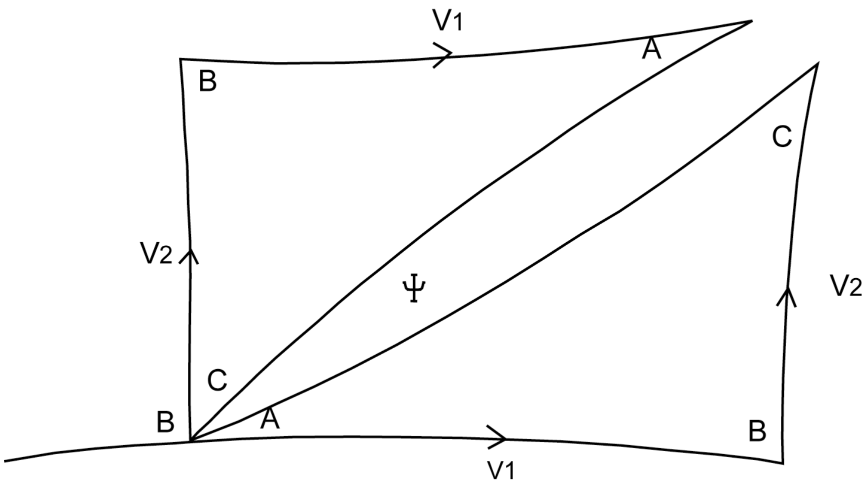


Fig: Sommerfeld's figure in hyperbolic form

Here there are two reversed congruent triangles. There is an angle Ψ between them because in hyperbolic geometry the angles of each triangle add up to be less than 180 degrees by the hyperbolic deficit D which is

$$D = \pi - (A + B + C) > 0$$

This is seen to be equal to angle Ψ in the diagram.

Vector Lorentz Transformations

Consider two reference frames S and S' with S' moving with velocity **v** relative to S. The differential form of the Lorentz transformation from S' to S and its reverse is

$$c\,dt = \gamma\,(c\,dt' + \mathbf{v}.\,d\mathbf{r}'/c)$$
$$d\mathbf{r} = \gamma\,\mathbf{v}\,dt' + d\mathbf{r}' + (\gamma - 1)\mathbf{v}\,(\mathbf{v}.d\mathbf{r}')/v^2$$

$$c\,dt' = \gamma\,(c\,dt - \mathbf{v}.\,d\mathbf{r}/c)$$
$$d\mathbf{r}' = -\gamma\,\mathbf{v}\,dt + d\mathbf{r} + (\gamma - 1)\mathbf{v}(\mathbf{v}.d\mathbf{r})/v^2$$

Here γ is $1/\sqrt{(1- v^2/c^2)}$. The equations differ only by the sign of **v**.
With partitioned matrices the equations can be written concisely e.g.

$$\begin{bmatrix} cdt \\ \mathbf{dr} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\mathbf{v}^T/c \\ \gamma\mathbf{v}/c & I + (\gamma-1)\mathbf{nn}^T/c^2 \end{bmatrix} \begin{bmatrix} cdt' \\ \mathbf{dr'} \end{bmatrix}$$

$$\mathbf{v} = [v_1, v_2, v_3]^T, \quad \mathbf{n} = \mathbf{v}/v$$

Using c dt instead of dt makes the matrix symmetric and its inverse for reverse transformation equal to L(-**v**). Bold letters are used for 3x1 column vectors for emphasis and ^{T} means transpose. The Lorentz matrix here will be denoted by L(**v**) is most appropriately called a *Lorentz translation matrix* being an analogue of Euclidean translations In full

$$\begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v_1/c & \gamma v_2/c & \gamma v_3/c \\ \gamma v_1/c & 1+(\gamma-1)n_1^2 & (\gamma-1)n_1n_2 & (\gamma-1)n_1n_3 \\ \gamma v_2/c & (\gamma-1)n_2n_1 & 1+(\gamma-1)n_2^2 & (\gamma-1)n_2n_3 \\ \gamma v_3/c & (\gamma-1)n_3n_1 & (\gamma-1)n_3n_2 & 1+(\gamma-1)n_3^2 \end{bmatrix} \begin{bmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{bmatrix}$$

This matrix is often written differently in the literature and called *boost* - an inappropriate name in the writer's opinion, especially in view of the results of the present paper.

The Notation of Mocanu and Ungar

Division of the Lorentz equations gives the velocities **u**, **u**' of a moving point relative to the two frames.

$$\mathbf{u} = \frac{\gamma\,\mathbf{v} + \mathbf{u}' + (\gamma - 1)\,\mathbf{v}(\mathbf{v}.\mathbf{u}')/v^2}{\gamma\,(1 + \mathbf{u}'.\mathbf{v}/c^2)}$$

$$\mathbf{u}' = \frac{-\gamma\,\mathbf{v} + \mathbf{u} + (\gamma - 1)\mathbf{v}(\mathbf{v}.\mathbf{u})/v^2}{\gamma\,(1 - \mathbf{u}.\mathbf{v}/c^2)}$$

Here a notation of Mocanu 1986 and Ungar 1989 may conveniently be used, putting for any two velocities $\mathbf{v}_1$, $\mathbf{v}_2$

$$\mathbf{v}_1 \circ \mathbf{v}_2 = \frac{\gamma_1\mathbf{v}_1 + \mathbf{v}_2 + (\gamma_1 - 1)\,\mathbf{v}_1(\mathbf{v}_1.\mathbf{v}_2)/v_1^2}{\gamma_1(1 + \mathbf{v}_1\,\mathbf{v}_2/c^2)}$$

In this γ_1 refers to $\mathbf{v}_1$. The equations then take the simple form.

$$\mathbf{u} = \mathbf{v} \circ \mathbf{u}' \qquad\qquad \mathbf{u}' = (-\mathbf{v}) \circ \mathbf{u}$$

The second expression gives relative velocity of the moving frames and coincides with Fock's relative velocity. Generally, Fock's relative velocity of two moving points P_1, P_2 is given by

$$(-\mathbf{v}_1) \circ \mathbf{v}_2 = \frac{-\gamma_1\mathbf{v}_1 + \mathbf{v}_2 + (\gamma_1 - 1)\,\mathbf{v}_1(\mathbf{v}_1.\mathbf{v}_2)/v_1^2}{\gamma_1(1 - \mathbf{v}_1\,\mathbf{v}_2/c^2)}$$

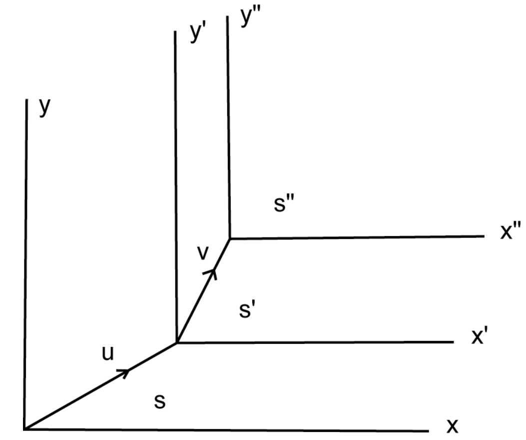
Non-commutativity and non-associativity: It may be seen that

$$\mathbf{v}_1 \circ \mathbf{v}_2 \neq \mathbf{v}_2 \circ \mathbf{v}_1 \qquad\qquad \mathbf{v}_1 \circ (\mathbf{v}_2 \circ \mathbf{v}_3) \neq (\mathbf{v}_1 \circ \mathbf{v}_2) \circ \mathbf{v}_3,$$

The resulting complex algebraic structure has been discussed by Ungar.

The Mocanu paradox

Mocanu (1986) introduced three frames S, S', S'' moving as shown and observed that the velocity **vou** of S'' relative to S is not the negative of the velocity **uov** of S relative to S''. He said that this contradicted 'Einstein's velocity reciprocity principle'.

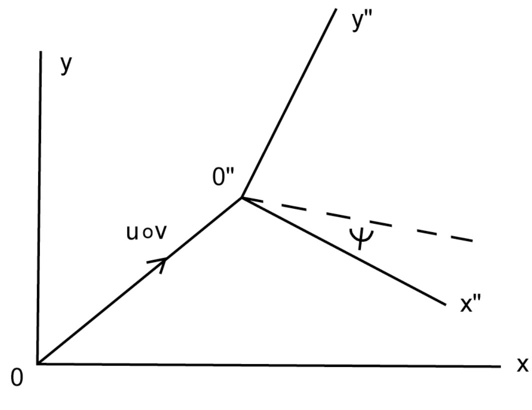


We make the following comments:

(a) The quoted principle refers to coordinate transformations not velocities. Coordinate transformation has symmetry apart from a difference in the sign of velocity but the inverse relation between velocities does not have this symmetry as was already clear in the simple case treated by Einstein in 1905.

(b) As already seen, Sommerfeld (1908) showed commutativity fails in the two dimensional case and was seen to be explainable by hyperbolic geometry. The present case, also two-dimensional in the **u, v** plane, will have a similar explanation.

Ungar (1989) said Mocanu's paradox can be explained by Thomas rotation of the axes of S'' relative to S due to multiplication of the Lorentz matrices for **v** and **u** as in the figure.



Our comments are

(a) As in the Sommerfeld 1908 case discussed before it is not correct to use matrix multiplication for velocity composition. Doing so introduces additional rotation equal to the Thomas rotation.

(b) Matrix multiplication can be used for composition of four-velocities only in the modified sense described below where the Thomas rotation does not cause a difficulty.

Use of Four-Velocity Vectors

Suppose the transformation equations between the frames are written in matrix form, e.g.

$$\begin{bmatrix} cdt \\ \mathbf{dr} \end{bmatrix} = [L(\mathbf{v})] \begin{bmatrix} cdt' \\ \mathbf{dr'} \end{bmatrix}$$

$$\begin{bmatrix} cdt' \\ \mathbf{dr}' \end{bmatrix} = [L(-\mathbf{v})] \begin{bmatrix} cdt \\ \mathbf{dr} \end{bmatrix}$$

Then it is possible to divide by the invariant time dτ and get the corresponding transformation and its inverse of the Minkowski vector.

$\mathbf{U} = L(\mathbf{v})\,\mathbf{U}'$ $\qquad\qquad$ $\mathbf{U}' = L(\mathbf{v})^{-1}\,\mathbf{U} = L(-\mathbf{v})\,\mathbf{U}$

Symmetry is here preserved between forward and backward equations - unlike the unsymmetrical relation for ordinary velocities. These equations may be regarded as adding or subtracting the contribution of the velocity **v** to the four vector of the reference frame.

However for three observers O, O', O", the transitivity property fails to hold because here it is true that the repetition of the multiplication of Lorentz matrices results in appearance of the Thomas rotation:

There is however a solution (see next section).

Matrix Definition of Relative Velocity

Suppose there are two moving points P_1, P_2 whose relative velocity we wish to define for an observer O. The velocity 4 vectors referred to observer O are

$$U^{(1)} = L(\mathbf{v}_1)\, U^{(0)} \qquad\qquad U^{(2)} = L(\mathbf{v}_2)\, U^{(0)}$$

So

$$U^{(2)} = L(\mathbf{v}_2)\, L(\mathbf{v}_1)^{-1}\, U^{(1)}$$

The velocity of P_2 relative to P_1 is consequently associated with the matrix

$$\Lambda_{2/1} = L(\mathbf{v}_2)\, L(\mathbf{v}_1)^{-1} = L(\mathbf{v}_2)\, L(-\mathbf{v}_1)$$

Note that this matrix is a general Lorentz transformation and is a translation only in the case when $\mathbf{v}_1$ is zero. Being a product of two Lorentz translations it takes the form

$$\Lambda_{2/1} = R\, L(\mathbf{v}_{2/1}) = -\, L(\mathbf{v}_{1/2})\, R$$

Here the translation matrices $L(\mathbf{v}_{2/1})$, $L(\mathbf{v}_{1/2})$ are defined by this equation. The reverse relative velocity matrix is

$$\Lambda_{1/2} = L(\mathbf{v}_1)\, L(\mathbf{v}_2)^{-1} = (\Lambda_{2/1})^{-1}$$

And this will have representation

$$\Lambda_{1/2} = L(\mathbf{v}_{1/2})\, R = -\, R\, L(\mathbf{v}_{2/1})$$

Transitivity of relative velocity matrices: Let there be three points P_1, P_2, P_3 moving with velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ relative to the observer O. The relative velocity matrices are

$$\Lambda_{2/1} = L(\mathbf{v}_2)\, L(\mathbf{v}_1)^{-1}$$
$$\Lambda_{3/2} = L(\mathbf{v}_3)\, L(\mathbf{v}_2)^{-1}$$
$$\Lambda_{3/1} = L(\mathbf{v}_3)\, L(\mathbf{v}_1)^{-1}$$

There follows the transitivity (associativity) law for sequentially connected relative velocities.

$$\Lambda_{3/1} = \Lambda_{3/2}\, \Lambda_{2/1}$$

Result of change of observer: Although the matrices L($\mathbf{v}_2$), L($\mathbf{v}_1$) are defined relative to the observer O, the product $\Lambda_{2/1}$ has a simple transformation law on change of observer which gives it an independence from the observer. Thus for two observers O, O* we may show that there are rotation matrices R_1, R_2 associated with moving points P_1, P_2 so that

$$\Lambda_{2/1}{}^* = R_2{}^{-1}\Lambda_{2/1}\, R_1$$

We find correspondingly for 3 observers

$$\Lambda_{3/1}{}^* = \Lambda_{3/2}{}^*\, \Lambda_{2/1}{}^*$$

$$= (R_3{}^{-1}\Lambda_{3/2}\, R_2)\, (R_2{}^{-1}\Lambda_{2/1}\, R_1)$$

$$= R_3{}^{-1}\Lambda_{3/1}\, R_1$$

This verifies invariance of the transitivity property for a change of observer.

Moving frames of reference: If system S' moves with velocity **v** relative to S, consider the motion of a point P moving with velocities **u**, **u'** relative to S and S'. As seen by an observer at rest at the origin O of the S frame, the velocity of P relative to the origin O' in the S' frame is

$$\Lambda(\mathbf{u'}) = L(\mathbf{u})\, L(\mathbf{v})^{-1}$$

Now the velocity **u** will be defined by the equation

$$L(\mathbf{u'})\, L(\mathbf{v}) = R\, L(\mathbf{u})$$

R is here a rotation matrix. So the observed relative velocity of P is

$$\Lambda(\mathbf{u'}) = R^{-1}\, L(\mathbf{u'})$$

The transformation law then takes the form

$$L(\mathbf{u}) = \{R^{-1}\, L(\mathbf{u'})\}L(\mathbf{v}) = \Lambda(\mathbf{u'})\, L(\mathbf{v})$$

This implies that it is not possible to find L(**u**) by just multiplying Lorentz translation matrices L(**v**) and L(**u'**). While valid mathematically, the product would have no physical meaning. Multiplication must be done as explained here the resultant having no rotation and addition taking place in the usual way for vectors.

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