Single Postulate Special Theory of Relativity

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0. Abstract

The current paper derives a more general form of the special theory of relativity (STR). In the derivation of the more general special theory of relativity (GSTR) only one of the two axioms of the special theory of relativity will be used, the axiom of equivalence of inertial systems. To prove the correctness of the newly derived GSTR we will derive a generalized Lorentz transformation and we will re-explain a number of experiments that are generally accepted as tests of STR. The paper is a theoretical exercise in proving relativity by using only the principle of relativity, very similar to the essays by R.C. Tolman and J.Kunz[7-10].We will demonstrate that the standard experimental tests of relativity, namely the Michelson-Morley[3], the Kennedy-Thorndike[4] and the Ives-Stilwell[6] experiments are not sufficient in order to distinguish between STR and GSTR.

1. Introduction

We will start with a reprise of a thought experiment, as described in [1].

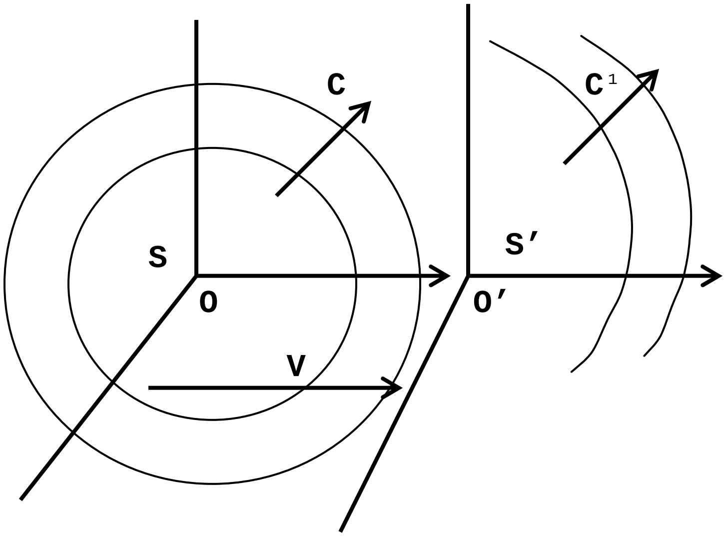

Figure 1. A thought experiment

Assume two inertial frames, S and S′ having the three axis in alignment. At t=0 the origins of the two inertial frames O and O′ coincide and S′ starts a rectilinear and uniform motion with the speed v along the axis Ox. An observer placed in O considers the metric :

$$x^2+y^2+z^2-u^2=0 \qquad \text{where } u=ct \tag{1.1}$$

An observer placed in O′ considers the metric :

$$x'^2+y'^2+z'^2-u'^2=0 \qquad \text{where } u'=c't' \tag{1.2}$$

As described in the footnote of the original 1905 Einstein paper[1] "The equations of the Lorentz transforms may be more simply deduced directly from the condition that in virtue of those equations the relation $x^2+y^2+z^2=(ct)^2$ shall have as a consequence the second relation $\xi^2+\eta^2+\zeta^2=(c\tau)^2$". By the time Einstein had written these lines, he had already set c′=c through his clock synchronization condition. If we accept that to be true from Einstein's footnote it follows that from the equality:

$$x^2+y^2+z^2-u^2= x'^2+y'^2+z'^2-u'^2 \tag{1.3}$$

we will be able to derive a more general form of the Lorentz transforms in the absence of the second postulate.

The relationship between the unprimed and the primed variables must be linear:

$$\begin{aligned} x' &= a_1x+b_1y+d_1z+e_1u \\ y' &= a_2x+b_2y+d_2z+e_2u \\ z' &= a_3x+b_3y+d_3z+e_3u \\ u' &= a_4x+b_4y+d_4z+e_4u \end{aligned}$$

All the 16 coefficients are functions of v but 9 of them will be proven to be null.

For x′=0 x=vt for any y and z. This means that x′ is independent of y and z, therefore $b_1=d_1=0$.

For y′=0 y=0 for any x,z and u, meaning that $a_2=d_2=e_2=0$.

For z′=0 z=0 for any x,y and u, meaning that $a_3=b_3=e_3=0$.

The relationship between y′ and y is identical with the relationship between z′ and z, resulting indo $b_2=d_3=q$.

$$\begin{aligned} x' &= a_1x+e_1u \\ y' &= qy \\ z' &= qz \\ u' &= a_4x+b_4y+d_4z+e_4u \end{aligned} \tag{1.3a}$$

The relationship between y and y′ needs to be the same as the relationship between y′ and y:

y=qy′ resulting into $y=q^2y$ which brings about $b_2=d_3=q=1$.

For x′=0 corresponds x=vt=βu where β=v/c. Therefore, $x'=a_1x+e_1u$ becomes :

$0=a_1\beta u+e_1 u$ imposing immediately $e_1=-a_1\beta$ resulting into:

$$x'=a_1(x-\beta u)$$

Introducing the above back into (1.3) we obtain:

$$x^2+y^2+z^2-u^2=(a_1{}^2-a_4{}^2)x^2+(1-b_4{}^2)y^2+(1-d_4{}^2)z^2+(a_1{}^2\beta^2-e_4{}^2)u^2$$
$$-2[a_4b_4xy+a_4d_4xz+(a_4e_4+a_1{}^2\beta)xu]-2(b_4d_4yz+b_4e_4yu+d_4e_4zu)$$

Through identification of the coefficients of like variables it follows immediately that:

$$a_1{}^2=1+a_4{}^2$$
$$-1=a_1{}^2\beta^2-e_4{}^2$$
$$b_4=d_4=0$$
$$a_4e_4+a_1{}^2\beta=0$$

Eliminating e_4 we obtain immediately that : $a_1=\frac{1}{\sqrt{1-\beta^2}}$

$$e_4=a_1=\frac{1}{\sqrt{1-\beta^2}}$$

Finally $a_4=-a_1\beta=-\frac{\beta}{\sqrt{1-\beta^2}}$

The final form of the transformations (1.3a) becomes:

$$x'=\frac{x-\beta u}{a} \tag{1.4}$$

$$y'=y$$

$$z'=z$$

$$u'=\frac{u-\beta x}{a} \tag{1.4a}$$

where

$$a=\sqrt{1-\beta^2}$$

The transformations (1.4) are the generalized Lorentz (GL) transformations. GSTR is based on only one axiom and replaces the standard Lorentz transformations with GL as its fundamental transformations.

By adding Einstein’s second postulate : c’=c we recover the standard Lorentz transformation. Transformations (1.4) can be inverted by eliminating x and u between (1.4) and (1.4a) producing:

$$x=\frac{x'+\beta u'}{a} \tag{1.5}$$

y=y′

z=z′

$$u=\frac{u'+\beta x'}{a}$$

Note the absence of any term of the form a′, this is natural given the way we derived (1.5) via direct algebraic inversion of (1.4).

Assume now that there is a time measuring device at rest with respect to S′, located at position x′. When this device measures t'_1 , the perceived time in S is given by (1.5) as :

$$ct_1=\frac{c't'_1+(v/c)x'}{a}$$

When this device measures t'_2 , the perceived time in S is given by (1.5) as :

$$ct_2=\frac{c't'_2+(v/c)x'}{a}$$

The time delta is then :

$$c\Delta t=(c'\Delta t')/a \qquad (1.5a)$$

the time dilation. While STR deals with time (t) dilation , GSTR considers the quantity ct as dilating. This formula can also be viewed as a light wavelength dilation: $\Delta\lambda=\Delta\lambda'/a$

Let dx/dt=w be the speed of an object moving along the axis Ox in S. To obtain the speed w′=dx′/dt′ of the same object as observed from S′ we will calculate w=dx/dt by using the first formula in the set (1.5) :

$$a\frac{dx}{dt}=\frac{dx'}{dt}+c'\frac{v}{c}\frac{dt'}{dt}=(\frac{dx'}{dt'}+c'\frac{v}{c})\frac{dt'}{dt}=(w'+c'\frac{v}{c})\frac{dt'}{dt}$$

$$aw=(w'+c'\frac{v}{c})\frac{dt'}{dt} \qquad (1.6)$$

From the last formulas in the set (1.4) it follows that:

$$a*c'\frac{dt'}{dt}=c-\frac{v}{c}\frac{dx}{dt}=c-\frac{vw}{c} \qquad (1.7)$$

Substituting (1.7) in (1.6) we obtain:

$$\frac{w'}{c'}=\frac{w/c-v/c}{1-v/c*w/c} \qquad (1.8)$$

For c′=c we recover the well known formula of speed transformation in standard special relativity.

By inverting (1.8) we can also obtain:

$$\frac{w}{c}=\frac{w'/c'+v/c}{1+v/c*w'/c'} \qquad (1.9)$$

Using the notation $w_r=\frac{w}{c}$ formulas (1.8) and (1.9) can be rewritten as:

$$w_r'=\frac{w_r-v_r}{1-w_r v_r} \qquad (1.10)$$

and respectively

$$w_r=\frac{w_r'+v_r}{1+w_r'v_r} \qquad (1.11)$$

Exactly like the speed transformations in STR, the formulas (1.10) and (1.11) form a group.

In the following paragraphs we will attempt to find practical experiments that could decision between the standard Lorentz transformations and the generalized ones.

2. The Tolman experiment[2]

$$\frac{w}{c}=\frac{w'/c'+v/c}{1+v/c*w'/c'} \qquad (2.1)$$

The speed of light as measured in system S is c and the speed of system S′ with respect with S is v. In S′ we consider two spheres of equal mass (as measured in S′) $m_1=m_2=m$. The two spheres move with the speeds w_1' and respectively w_2' (as measured in S′). Let’s assume $w_1'=-w_2'$. (2.2)

After the two spheres collide, since they have equal mass and equal and opposite speeds they will lie at rest for a short time with respect to S′, meaning that they will form a body of mass (m_1+m_2) that moves with the speed v :

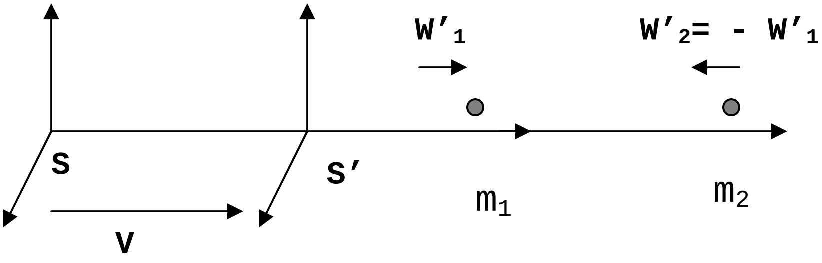

Figure 2. The Tolman experiment

As seen from S :

$$m_1*w_1+m_2*w_2=(m_1+m_2)v \qquad (2.3)$$

From (2.3) we obtain:

$$(m_1+m_2)v/c=m_1*w_1/c+m_2*w_2/c=m_1\frac{w_1'/c'+v/c}{1+v/c*w_1'/c'}+m_2\frac{w_2'/c'+v/c}{1+v/c*w_2'/c'} \qquad (2.4)$$

Substituting $w_2'=-w_1'$ in (2.4) we obtain:

$$\frac{m_1}{m_2}=\frac{1+w'_1/c'*v/c}{1-w'_1/c'*v/c} \qquad (2.5)$$

Let's assume $w_2'=-c'/c*v$. Substituting in (2.1) we get $w_2=0$, i.e. sphere 2 is at rest with respect to S (sphere 1 isn't).

$w_1'=-w_2'=c'/c*v$ implying $w_1=\frac{2v}{1+(v/c)^2}$

Substituting in (2.5) we obtain:

$$\frac{m_1[w_1]}{m_2[w_2]}=\frac{1+(v/c)^2}{1-(v/c)^2} \qquad (2.6)$$

Here $m_i[w_i]$ signifies that the mass m_i is dependent on the speed w_i.
We can rewrite (2.6) as :

$$m_1[2v/(1+(v/c)^2)]=m[0]*\frac{1+(v/c)^2}{1-(v/c)^2} \qquad (2.7)$$

If we use the notation : $2\,v/[1+(v/c)^2]=\varpi$
we obtain :

$$m_1[\varpi]=\frac{m[0]}{\sqrt{1-(\omega/c)^2}} \qquad (2.8)$$

which is exactly the formula of mass variation with speed in the special theory of relativity. The reasoning shown above leads us to believe that it is not possible to construct a dynamic mechanics based experiment that could differentiate between the standard and the non-standard Lorenz transformations.

3. Relativistic electrodynamics-the Lorentz force as a relativistic effect

Let P be a fixed point in the inertial frame S and let S' be an inertial frame moving with the speed v along the x axis. Let's calculate the partial derivative $\partial E_x/\partial t$:

$$\partial E_x/\partial t=\partial E_x/\partial x'*\partial x'/\partial t+\partial E_x/\partial y'*\partial y'/\partial t+\partial E_x/\partial z'*\partial z'/\partial t+\partial E_x/\partial t'*\partial t'/\partial t \qquad (3.1)$$

Since P is fixed x,y,z are fixed, from (1.4) we get :

$\partial x'/\partial t=-v/a$, $\partial y'/\partial t=\partial y/\partial t=0$, $\partial z'/\partial t=\partial z/\partial t=0$

$c'*\partial t'/\partial t=c/a$

Therefore:

$$a*\partial E_x/\partial t=c/c'*\partial E_x/\partial t'-v*\partial E_x/\partial x' \quad (3.2)$$

Let's calculate the partial derivative :

$$\partial E_x/\partial x'=\partial E_x/\partial x*\partial x/\partial x'+\partial E_x/\partial y*\partial y/\partial x'+\partial E_x/\partial z*\partial z/\partial x'+\partial E_x/\partial t*\partial t/\partial x'=$$

$$=1/a*\partial E_x/\partial x+v/(c^2*a)*\partial E_x/\partial t=1/a*(\partial E_x/\partial x+v/c^2)*\partial E_x/\partial t \quad (3.3)$$

Substituting (3.3) in (3.2) we obtain:

$$\partial E_x/\partial t=1/a*c/c'*\partial E_x/\partial t'-v/a^2*(\partial E_x/\partial x+v/c^2*\partial E_x/\partial t)$$

$$(1+(v/(ac)^2)*\partial E_x/\partial t=c/(ac')\partial E_x/\partial t'-v/a^2*\partial E_x/\partial x$$

$$\partial E_x/\partial t=ac/c'*\partial E_x/\partial t'-v*\partial E_x/\partial x \quad (3.4)$$

From Maxwell's laws we have :

$0=\partial E_x/\partial x+\partial E_y/\partial y+\partial E_z/\partial z$ i.e. $\partial E_x/\partial x=-(\partial E_y/\partial y+\partial E_z/\partial z)$.

Substituting in (3.4) we get:

$$\partial E_x/\partial t=ac/c'*\partial E_x/\partial t'+v*(\partial E_y/\partial y+\partial E_z/\partial z)$$

Also from Maxwell's laws we have:

$$\partial H_z/\partial y-\partial H_y/\partial z=\varepsilon_0*\partial Ex/\partial t=\varepsilon_0 ac/c'*\partial E_x/\partial t'+\varepsilon_0 v(\partial E_y/\partial y+\partial E_z/\partial z)$$

$$\varepsilon_0 ac/c'*\partial E_x/\partial t'=\partial/\partial y(H_z-\varepsilon_0 vE_y)-\partial/\partial z(H_y+\varepsilon_0 vE_z)=\partial/\partial y'(H_z-\varepsilon_0 vE_y)-$$

$$-\partial/\partial z'(H_y+\varepsilon_0 vE_z)$$

or:

$$a*\varepsilon_0 c/c'*\partial E_x/\partial t'=\partial/\partial y'(H_z-\varepsilon_0 vE_y)-\partial/\partial z(H_y+\varepsilon_0 vE_z) \quad (3.5)$$

The law $\partial H_z/\partial y-\partial H_y/\partial z=\varepsilon_0*\partial E_x/\partial t$ as written in S must be rewritten in system S'as:

$$\partial H'_z/\partial y'-\partial H'_y/\partial z'=\varepsilon_0*\partial E'_x/\partial t' \quad (3.6)$$

Comparing (3.6) with (3.5) we conclude that:

$$E'_x=E_x*c/c'$$
$$H'_z=(H_z-\varepsilon_0 vE_y)/a$$
$$H'_y=(H_y+\varepsilon_0 vE_z)/a \quad (3.7)$$

For c'=c we recover the standard transformation.

In a similar manner as above we derive that:

$$H'_x=H_x*c/c'$$
$$E'_z=(E_z+\upsilon_0 vH_y)/a$$
$$E'_y=(E_y-\upsilon_0 vH_z)/a \quad (3.8)$$

Let's assume that the frame S is at rest with respect to a magnetic field. Let's also assume that there is no electric field. In the frame S', according to (3.7) and (3.8) we will measure an electric field:

$E'_x=E_x=0$

$E'_y=-1/a^*\ u_0vH_z=-v/a^*B_z$

$E'_z=1/a^*\ u_0vH_y=+v/a^*B_y$

This electric field will exert an electrostatic force on a particle of charge q at rest with respect to S' : **F'**=q**E'**

$F'_x=0$

$F'_y=-qv/a^*B_z$

$F'_z=+qv/a^*B_y$ (3.9)

(3.9) is in perfect agreement with the special theory of relativity, the term c/c' has again "disappeared".

Let's try to see what happens when examining the non-homogenous Maxwell equations.

From $\rho/\varepsilon_0=\partial E_x/\partial x+\partial E_y/\partial y+\partial E_z/\partial z$ we obtain $\partial E_x/\partial x=-(\partial E_y/\partial y+\partial E_z/\partial z)+\rho/\varepsilon_0$

Substituting in (3.4) we get:

$\partial E_x/\partial t=ac/c'^*\partial E_x/\partial t'+v^*(\partial E_y/\partial y+\partial E_z/\partial z)-v\rho/\varepsilon_0$

Also from Maxwell's laws we have:

$\partial H_z/\partial y-\partial H_y/\partial z=J+\varepsilon_0{}^*\partial Ex/\partial t=J+\varepsilon_0 ac/c'^*\partial E_x/\partial t'+\varepsilon_0 v(\partial E_y/\partial y+\partial E_z/\partial z-$

$-\rho/\varepsilon_0)=\varepsilon_0 ac/c'^*\partial E_x/\partial t'+\varepsilon_0 v(\partial E_y/\partial y+\partial E_z/\partial z)-v\rho+J$

$a*\varepsilon_0 c/c'^*\partial E_x/\partial t'+J-v\rho=\partial/\partial y'(H_z-\varepsilon_0 vE_y)-\partial/\partial z(H_y+\varepsilon_0 vE_z)$ (3.10)

The law $\partial H_z/\partial y-\partial H_y/\partial z=J+\varepsilon_0{}^*\partial E_x/\partial t$ as written in S must be rewritten in system S' as:

$J'+\varepsilon_0{}^*\partial E'_x/\partial t'=\partial H'_z/\partial y'-\partial H'_y/\partial z'$ (3.11)

Comparing (3.11) and (3.10) we obtain :

$J'=(J-v\rho)/a$ (3.12)

$E'_x=E_x{}^*c/c'$

$H'_z=(H_z-\varepsilon_0 vE_y)/a$

$H'_y=(H_y+\varepsilon_0 vE_z)/a$

In a similar manner as above we derive that:

$H'_x=H_x{}^*c/c'$

$E'_z=(E_z+\upsilon_0 vH_y)/a$

$E'_y=(E_y-\upsilon_0 vH_z)/a$ (3.13)

Let's assume that the frame S is at rest with respect to a magnetic field. Let's also assume that there is no electric field. In the frame S', according to (3.12) and (3.13) we will measure an electric field:

$E'_x=E_x=0$

$E'_y=-1/a^*\ u_0vH_z=-v/a^*B_z$

$E'_z=1/a^*\ u_0vH_y=+v/a^*B_y$

This electric field will exert an electrostatic force on a particle of charge q at rest with respect to S′ : **F′**=q**E′**

$$F'_x=0$$
$$F'_y=-qv/a*B_z$$
$$F'_z=+qv/a*B_y \qquad (3.14)$$

Thus, (3.14) is in perfect agreement with the special theory of relativity, the term c/c′ has again “disappeared”.

4. The Michelson – Morley and the Kennedy-Thorndike Experiments[3,4,5]

Let’s consider the Earth, with the Michelson-Morley setup as the frame S′ in motion with the speed v with respect to another frame S. An observer at rest in S views the Michelson-Morley as follows: the light travels with the speed c along the rod of length l_2 , parallel with v. Because the mirror of the end of the rod recedes with the speed +v in one direction of light propagation and with speed –v in the other direction, the observer in S views the roundtrip time as :

$$t_2=\frac{l_2}{c+v}+\frac{l_2}{c-v}=\frac{2l_2}{ca^2} \qquad (4.1)$$

The rod perpendicular on v has the length l_1. In S′ $l_1=l_2$.
From S, the observer sees $(ct)^2=l_1^2+(vt)^2$ from where we obtain:

$$t=\frac{l_1}{ac}$$

The roundtrip time as seen from S is:

$$t_1=2*t=\frac{2l_1}{ac}=\frac{2l'_1}{ac} \qquad (4.2)$$

According to (1.4), as seen from S, $l_1=l_1'$ and $l_2=a*l_2'$
Therefore:

$$t_2=\frac{2l'_2}{ac} \qquad (4.3)$$

From the perspective of S,

$$t_2-t_1=\frac{2(l_2'-l_1')}{ac}\,. \qquad (4.4)$$

From (4.4) we get that, in S′,

$$t_2'-t_1'=a*c/c'*(t_2-t_1)=\frac{2(l_2'-l_1')}{c'}=0\,. \qquad (4.5)$$

Or, otherwise written :

$$1/a*\Delta\lambda'=\Delta\lambda=0 \qquad (4.6)$$

(4.5) explains the null result of the Michelson-Morley experiment as viewed from the frame S', the Earth.

In 1932, Kennedy and Thorndike devised a modified version of the Michelson-Morley experiment that relies on making the two arms of the interferometer as uneven in length as possible and on using the variation of the Earth speed in its revolution around the Sun. The Kennedy Thorndike experiment has been refined repeatedly in the past few years [4,5] Let the arms of the interferometer, as measured in S', be l'_1 and l'_2. From (4.5):

$$t_2' - t_1' = \frac{2(L'_2 - L'_1)}{c'} \qquad (4.7)$$

a relationship independent of the relative speed v, i.e. a result indistinguishable from the SRT result for the Kennedy-Thorndike experiment.

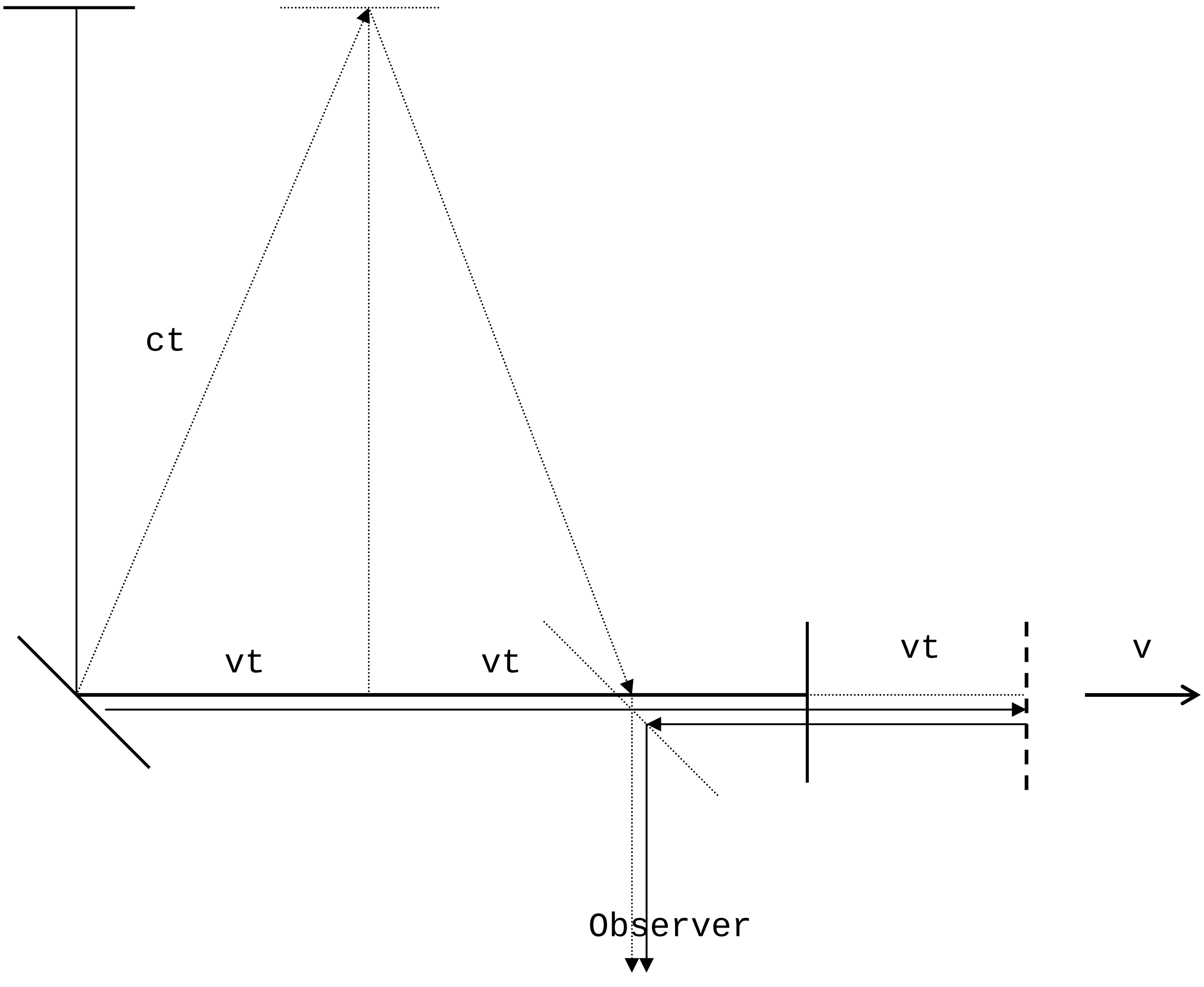

Figure 3. The Michelson – Morley experiment

5. The Ives-Stilwell Experiment[6]

The Ives-Stilwell experiment exploits the Doppler effect. Looking at figure 4, at time $t=0$ a light front is emitted from the source S that moves with the peed v along the line connecting observers S_1' and S_2'. At $t=0$ the source S is a distance d_1 from S_1'. The light front reaches S_1' at $t_1=d_1/c$.

At time $t=T$ (where T is the period of the light front as measured in an inertial frame tied to S) a second light front is emitted from S. S is now a distance $d_2=d_1+vT$ from S_1'. This second light front reaches S1′ at $t_2=T+d_2/c=T+t_1+vT/c$.

$$\Delta t=t_2-t_1=T+vT/c=T(1+v/c) \tag{5.1}$$

$$c\Delta t=cT(1+v/c)=\lambda(1+v/c) \tag{5.2}$$

From the introduction (section 1) we know that as viewed from S_1', the wavelength is perceived to be:

$$\lambda_1'=c'\Delta t'=\gamma\, c\, \Delta t=\gamma\,\lambda(1+v/c) \qquad \text{where } \gamma=1/a \tag{5.3}$$

In a similar fashion, we obtain that the wavelength perceived in S_2' due to S moving towards S_2' is:

$$\lambda_2'=\gamma\,\lambda(1-v/c) \tag{5.4}$$

S_1' and S_2' are part of the same inertial frame, therefore we can calculate the fringe shift between the front received in S_1' and the frame received and reflected in S_2' as:

$$\delta\lambda=\lambda_1'-\lambda_2'=\gamma\,\lambda(1+v/c)-\gamma\,\lambda(1-v/c)=2\gamma\,\lambda\ v/c \tag{5.5}$$

Once again, GSTR and STR predict the same result.

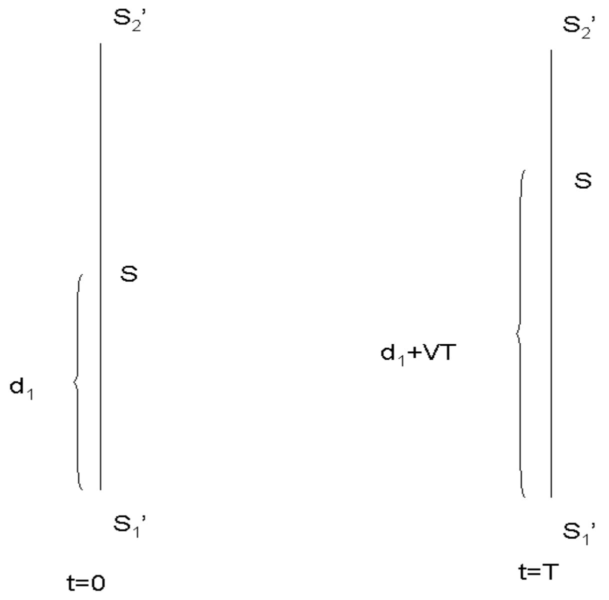

Figure 4. The Ives-Stilwell experiment

6. Conclusions

We have derived a generalization of STR, a new theory that is founded on one axiom only. GSTR is based on a generalized Lorentz transformation. We have used the generalized transformation to explain the results of well known experiments. We have demonstrated that the standard experimental tests of relativity, namely the Michelson-Morley[2], the Kennedy-Thorndike[4] and the Ives-Stilwell[6] experiments are not sufficient in order to distinguish between STR and GSTR underscoring the importance of very specific experiments that do prove that the light speed is indeed independent of the relative motion between source and the observer. Such experiments[11-17] are few and far apart in time, so it would be good to see some contemporary reenactments.

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Appendix A

We now turn our attention to chapter 3 ”Theory of the Transformation of Coordinates and Times from a Stationary System to another System in Uniform Motion of Translation Relatively to the Former”.[1] Throughout this derivation we will retain Einstein’s notation as much as possible.

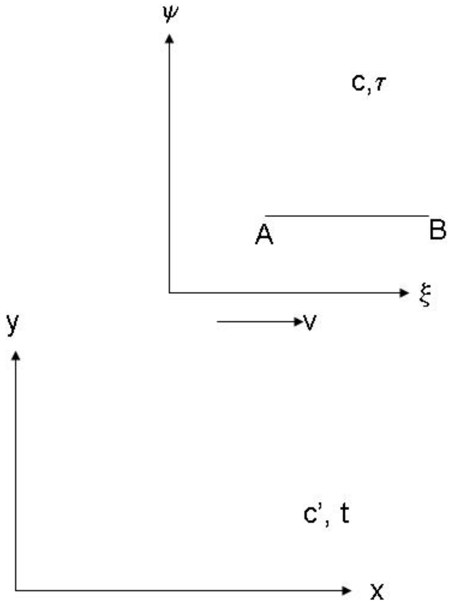

Fig.A1 Einstein’s setup

The setup described in fig. A1 comprises a rod AB with a light source at the A end and a mirror at the B end. The rod is at rest with respect to system (k). We will retain the term “system” used by Einstein instead of employing the more modern term of “frame”. System (k) and system (K) have their axes aligned and there is a movement with constant velocity v along the x-axis between the two systems. Referring to fig. 1 the system (k) measures time τ, light speed c as well as:

$$\tau_B-\tau_A=\tau'_A-\tau_B=r_{AB}/c \qquad (1.1)$$

System (K) “sees” a different length r'_{AB}, a different time t and a different speed for light c’.

During the light trip from A to B the mirror at the B end of the rod recedes with a speed v:

$$c't_+=r'_{AB}+\delta_+$$

$$\delta_+=vt_+$$

$$r'_{AB}+vt_+=c't_+$$

$$t_B-t_A=t_+=r'_{AB}/(c'-v) \qquad (1.2)$$

During the light trip from B to A the mirror at the B end of the rod recedes with a speed
–v:

$$c't_-=r'_{BA}+\delta_-$$

$$\delta_-=-vt_-$$

$$r'_{BA}-vt_-=c't_-$$

$$t'_A-t_B=t_-= r'_{AB}/(c'+v) \qquad (1.3)$$

It is sensible to assume that the length of the rod as “seen” by the system (K) is the same for both directions of the light ray:

$r'_{AB} = r'_{BA}$

Therefore:

$$(t_B-t_A)(c'-v)=(t'_A-t_B)(c'+v) \qquad (1.4)$$

This is the **new** synchronization rule. It can be rewritten, using Einstein’s notations as:

$$(c'-v)[t(\xi,\tau+\xi/c)-t(0,\tau)]=(c'+v)[t(0,\tau+2\xi/c)-t(\xi,\tau+\xi/c)] \qquad (1.5)$$

Let $t=t(\xi,\tau)$ be a continuous, two time differentiable function.

$$(c'-v)[t(\xi,\tau+\xi/c)-t(0,\tau)]=(c'+v)[t(0,\tau+2\xi/c)-t(\xi,\tau+\xi/c)] \qquad (1.6)$$

By applying Taylor expansion while stopping at the second order term we obtain the second order partial differential equation:

$$c'\frac{\partial t}{\partial \xi}-\frac{\partial t}{\partial \tau}\frac{v}{c}-\frac{v}{2}\left(\frac{\partial^2 t}{\partial \xi^2}\xi+\frac{\partial^2 t}{\partial \tau^2}\frac{\xi}{c^2}+2\frac{\partial^2 t}{\partial \xi \partial \tau}\frac{\xi}{c}\right)=0 \qquad (1.7)$$

Using the notation $u=c't$ and $U=c\tau$ the equation becomes:

$$\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial U}\frac{v}{c'}-\frac{v}{2}\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\xi}{c'^2}+\frac{\partial^2 u}{\partial U^2}\frac{\xi}{c'^2}+2\frac{\partial^2 u}{\partial \xi \partial U}\frac{\xi}{c'^2}\right)=0 \qquad (1.8)$$

For $v \ll c'$, since the term v/c'^2 is very close to 0, the equation (1.8) reduces to a simple, first degree equation:

$$\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial U}\frac{v}{c'}=0 \text{ with the immediate solution } u(\xi,U)=b\left(U+\frac{v}{c'}\xi\right) \qquad (1.9)$$

where b is a parameter that will be determined later.
If v is very close to c’, we can no longer neglect the terms in the second degree derivatives. By taking v/c’=A and $v/2c'^2$=B (where B=A/2c’) the equation becomes:

$$\frac{\partial u}{\partial \xi} - A\frac{\partial u}{\partial U} - B\xi(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial U^2} + 2\frac{\partial^2 u}{\partial \xi \partial U}) = 0 \qquad (1.10)$$

We will try to find the solution u=u(ξ,U) with the boundary condition u(0,0)=0.
We expect that a solution will be of the form: u(ξ,U)=f(ξ)+g(U), f(0)=0 and g(0)=0
Then, the equation becomes:

$$\frac{\partial f}{\partial \xi} - A\frac{\partial g}{\partial U} - B\xi(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 g}{\partial U^2}) = 0 \qquad (1.11)$$

We can rewrite the equation as:

$$\frac{\partial f}{\partial \xi} - B\xi\frac{\partial^2 f}{\partial \xi^2} = A\frac{\partial g}{\partial U} + B\xi\frac{\partial^2 g}{\partial U^2} \qquad (1.12)$$

The above implies immediately that $\frac{\partial^2 g}{\partial U^2} = 0$, resulting into. g(U)=bU
Then:

$$B\xi\frac{\partial^2 f}{\partial \xi^2} - \frac{\partial f}{\partial \xi} + Ab = 0 \qquad (1.13)$$

Let’s try f(ξ)=q ξ, then –q+Ab=0, i.e. q=Ab and f(ξ)=Abξ

$$u(\xi,U)=bU+Ab\xi = b(U + \frac{v}{c'}\xi) \qquad (1.14)$$

The fact that Einstein stopped his Taylor expansion at the first order term was criticized in the past as a tacit introduction of a “third” postulate in[1], namely the postulate of linear dependency. As we could see from the above derivation, even if we consider the higher order terms in the expansion, the resultant Lorentz transformations are “stubbornly” linear, validating Einstein’s original intuition.

There is no reason to “dream up” new transforms that employ more complicated functions.

In order to determine the parameter b we assume that the rod in system (k) gets rotated 90 degrees. The whole setup becomes very similar with the setup of the Michelson Morley experiment. While the ray of light bounces from A to B and back, the observer in (K) “sees” it running the path seen in fig.A2:

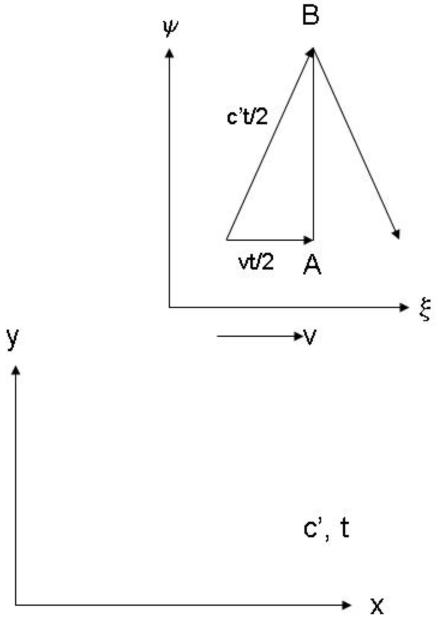

Fig.A2 Modified setup

$$(\frac{c'\Delta t}{2})^2-(\frac{v\Delta t}{2})^2=L^2 \qquad (1.15)$$

$$\Delta t=\frac{2L}{\sqrt{c'^2-v^2}} \qquad (1.16)$$

$$\Delta u = c'\Delta t = \frac{2L}{\sqrt{1-\frac{v^2}{c'^2}}} \tag{1.17}$$

$$\Delta U = c\Delta\tau = 2L \tag{1.18}$$

$$\Delta u = \frac{\Delta U}{\sqrt{1-\frac{v^2}{c'^2}}} \tag{1.19}$$

Since u(ξ,U)=$b(U+\frac{v}{c'}\xi)$ it follows that Δu=bΔU implying that

$$b = \frac{1}{\sqrt{1-\frac{v^2}{c'^2}}} \tag{1.20}$$

Now we are ready to derive the transformation x=x(ξ,U). Referring again to fig.1:

$$(t'_A - t_B) + (t_B - t_A) = \frac{r'_{AB}}{c'-v} + \frac{r'_{AB}}{c'+v} = \frac{2c'r'_{AB}}{c'^2 - v^2} \tag{1.21}$$

$$\Delta u = c'\Delta t = \frac{2c'^2 r'_{AB}}{c'^2 - v^2} \tag{1.22}$$

$$2r_{AB} = c\Delta t = \Delta U = \Delta u\sqrt{1-\frac{v^2}{c'^2}} \tag{1.23}$$

$$2r_{AB} = c\Delta t = \Delta U = \Delta u\sqrt{1-\frac{v^2}{c'^2}} = \frac{2r'_{AB}}{\sqrt{1-\frac{v^2}{c'^2}}}$$

(1.24)

$$r'_{AB} = r_{AB}\sqrt{1-\frac{v^2}{c'^2}}$$

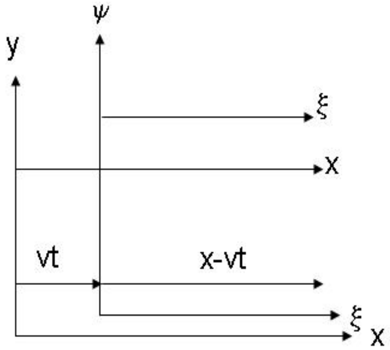

Fig.A3 x=x(ξ,U) calculation

Referring now to fig.A3 we can write:

$$x - vt = \xi\sqrt{1-\frac{v^2}{c'^2}}$$

$$\xi\sqrt{1-\frac{v^2}{c'^2}} = x - \frac{v}{c'}(c't) = x - \frac{v}{c'}u = x - \frac{v}{c'}b(U + \frac{v}{c'}\xi)$$

$$x = \xi\sqrt{1-\frac{v^2}{c'^2}} + \frac{v}{c'}\frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(U + \frac{v}{c'}\xi) = \frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(\xi + \frac{v}{c'}U) \qquad (1.25)$$

$$x(\xi,U) = \frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(\xi + \frac{v}{c'}U)$$

The generalized Lorentz transformations are :

$$x(\xi,U)=\frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(\xi+\frac{v}{c'}U)$$

$$u(\xi,U)=\frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(U+\frac{v}{c'}\xi) \qquad (1.26)$$

$$y=\psi$$

$$z=\zeta$$

The equations can be algebraically inverted in order to obtain the inverse transforms:

$$\xi(x,u)=\frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(x-\frac{v}{c'}u)$$

$$U(x,u)=\frac{1}{\sqrt{1-\frac{v^2}{c'^2}}}(u-\frac{v}{c'}x) \qquad (1.27)$$

$$\psi=y$$

$$\zeta=z$$

Finally, as in the original paper let’s calculate:

$$\begin{aligned} x^2-u^2 &= b^2[(\xi+\frac{v}{c'}U)^2-(U+\frac{v}{c'}\xi)^2]=b^2[\xi^2+(\frac{v}{c'})^2U^2-U^2-(\frac{v}{c'})^2\xi^2] \\ &= b^2(1-\frac{v^2}{c'^2})(\xi^2-U^2)=\xi^2-U^2 \end{aligned} \qquad (1.28)$$